

# FUZZY TRANSFORMATIONS AND EXTREMALITY OF GIBBS MEASURES FOR THE POTTS MODEL ON A CAYLEY TREE

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**ABSTRACT.** We continue our study of the full set of translation-invariant splitting Gibbs measures (TISGMs, translation-invariant tree-indexed Markov chains) for the  $q$ -state Potts model on the Cayley tree. In our previous work [10] we gave the full description of the TISGMs, and showed in particular that at sufficiently low temperatures their number is  $2^q - 1$ . In this paper we find some regions for the temperature parameter ensuring that a given TISGM is (non-)extreme in the set of all Gibbs measures. In particular we show the existence of a temperature interval for which there are at least  $2^{q-1} + q$  extremal TISGMs. For the Cayley tree of order two we give explicit formulae and some numerical values.

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## 1. INTRODUCTION

For the  $q$ -state Potts model on a Cayley tree of order  $k \geq 2$  it is known for a long time that at sufficiently low temperatures there are at least  $q+1$  translation-invariant Gibbs measures which are also tree-indexed Markov chains [3], [4]. Such translation-invariant tree-indexed measures are equivalently called translation-invariant splitting Gibbs measures (TISGMs). The  $q+1$  well-known measures mentioned above are obtained as infinite-volume limits with either the  $q$  boundary conditions of homogeneous spin-configurations, or the free boundary condition. While the  $q$  measures with homogenous boundary conditions are always extremals in the set of all Gibbs-measures, the free boundary condition measure is an extremal Gibbs measure only in an intermediate temperature interval below the transition temperature, and loses its extremality for even lower temperatures. The non-trivial problem to determine the precise transition value is also known as the reconstruction problem in information theoretic language. While the Kesten-Stigum bound gives a sufficient (but not necessary bound) in terms of second largest eigenvalue of the transition matrix of the chain (and hence the temperature parameter) which ensures non-extremality, the opposite problem, namely to ensure extremality in terms of bounds on parameters, is more difficult, and the known methods usually do not lead to sharp results for tree indexed Markov chains. This is true even in simple cases like the general asymmetric binary channel (or Ising model in a magnetic field). For the Potts model in

zero field it was proved in [19] that the Kesten-Stigum bound for the open boundary condition measure is not sharp for  $q \geq 5$ . For some numerical investigations see [14].

For other results related to the Potts model on Cayley trees see [6], [9], [10], [18] and the references therein.

Now, in [10] all TISGMs (tree-indexed Markov chains) for the Potts model are found on the Cayley tree of order  $k$ , and it is shown that at sufficiently low temperatures their number is  $2^q - 1$ . For the binary tree we gave explicit formulae for the critical temperatures and the possible TISGMs.

The analysis was based on the classification of translation-invariant boundary laws which are in one-to-one correspondence with the TISGMs. Recall that boundary laws are length- $q$  vectors with positive entries which satisfy a non-linear fixed-point equation (tree recursion), and a given boundary law defines the transition matrix of the corresponding Markov chain.

While the fact that these measures can never be nontrivial convex combinations of each other is almost automatic (see [10]) it is not clear whether and when they are extremals in the set of all Gibbs measures, including the non-translation invariant Gibbs measures. In particular, from [10] it was not clear yet, whether the complete set of TISGMs contained any new extremal Gibbs measures beyond the known  $q + 1$  measures, or whether the new TISGMs would all be non-extremal Gibbs measures. It is the purpose of this paper to give some answers and find some regions of parameters where a TISGM is (non-)extreme. To ask immediately for precise transition values would be too much to ask in view of the known difficulty of the aforementioned reconstruction problem already for the open boundary condition measure.

The paper is organized as follows. Section 2 contains preliminaries (necessary definitions and facts). All of our analysis relies heavily on the following useful fact: While the full permutation symmetry of the free boundary condition measure is lost in general, all the  $q \times q$  transition matrices which arise in the description of the TISGMs possess a  $2 \times 2$  block-structure. All possible sizes of blocks can appear, and labels within the blocks are equivalent. This corresponds to a decomposition of the  $q$  spin-labels into two classes  $1'$  and  $2'$ , one of  $m$  elements, the other one of  $q - m$  elements, with  $m \leq q/2$ . Such a structure invites the study of the associated fuzzy map (see [7], [9]) which identifies the spin-values within the classes, and which maps the initial Markov chain to a coarse-grained or fuzzy Markov chain with spin labels  $1', 2'$ . It is interesting to note that this Markov chain can be interpreted as a splitting Gibbs measure for an effective Ising Hamiltonian, and that this Hamiltonian is independent of the choice of the initial Gibbs measure (within the class indexed by  $m$ ). Such a result, namely the independence of the Hamiltonian for a transformed (renormalized) Gibbs measure of the phase, if this renormalized Hamiltonian exists at all, is known to be true for lattice models [1, 11] with proofs which are based on the variational principle and hence do not directly apply to trees.

Section 3 is devoted to this fuzzy transformation of the Potts model to an Ising model with an external field. There we study relations between eigenvalues of the

relevant transition matrix given by the Potts model and by the corresponding fuzzy Potts model. We derive conditions for extremality and non-extremality for the corresponding coarse-grained Markov chain. We note that non-extremality of the coarse-grained chain already implies non-extremality of the original chain.

Section 4 is related to non-extremality conditions of a TISGM (for the original model), where we check the Kesten-Stigum condition (based on the second largest eigenvalue). We give explicit formulas for critical parameters for the Kesten-Stigum condition to hold. This provides us with sufficient conditions for the temperature to see non-extremality. We cannot expect these conditions to be sharp in general, since the Kesten-Stigum bound is not sharp in most cases, but often the Kesten-Stigum bound is numerically not very far off.

Section 5 deals with extremality conditions for a TISGM. There are various approaches in the literature to sufficient conditions for extremality which can be reduced to a finite-dimensional optimization problem based only on the transition matrix, see the percolation method of [13], [15], the symmetric entropy method of [2], or for the binary asymmetric channel the readily available bound of Martin [12]. In this paper we employ the approach of [13], where the non-trivial part is the estimate of their constant  $\gamma$  (controlling "percolation from outside to inside") for the case of our transition matrices. As in a particular temperature region there are many TISGMs which have different transition matrices which all lie in a neighborhood of transition matrix of the free measure, and we know that extremality holds for the latter, a continuity argument in particular provides us with a lower bound on the number of extremal TISGMs which have to occur.

## 2. PRELIMINARIES

*Cayley tree.* The Cayley tree  $\Gamma^k$  of order  $k \geq 1$  is an infinite tree, i.e., a graph without cycles, such that exactly  $k+1$  edges originate from each vertex. Let  $\Gamma^k = (V, L)$  where  $V$  is the set of vertices and  $L$  the set of edges. Two vertices  $x$  and  $y$  are called *nearest neighbors* if there exists an edge  $l \in L$  connecting them. We will use the notation  $l = \langle x, y \rangle$ . A collection of nearest neighbor pairs  $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \dots, \langle x_{d-1}, y \rangle$  is called a *path* from  $x$  to  $y$ . The distance  $d(x, y)$  on the Cayley tree is the number of edges of the shortest path from  $x$  to  $y$ .

For a fixed  $x^0 \in V$ , called the root, we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^n W_m$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

the set of *direct successors* of  $x$ .

*Potts model.* We consider the Potts model on a Cayley tree, where to each vertex of the tree a spin variable is assigned which takes values in the local state space  $\Phi := \{1, 2, \dots, q\}$ .

The (formal) Hamiltonian of Potts model is

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \delta_{\sigma(x)\sigma(y)}, \quad (2.1)$$

where  $J \in \mathbb{R}$  is a coupling constant,  $\langle x, y \rangle$  stands for nearest neighbor vertices and  $\delta_{ij}$  is the Kroneker's symbol:

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j. \end{cases}$$

In the present paper we only consider the case of ferromagnetic interaction  $J > 0$ .

*Splitting Gibbs measure.* Define a finite-dimensional distribution of a probability measure  $\mu$  in the volume  $V_n$  as

$$\mu_n(\sigma_n) = Z_n^{-1} \exp \left\{ -\beta H_n(\sigma_n) + \sum_{x \in W_n} \tilde{h}_{\sigma(x), x} \right\}, \quad (2.2)$$

where  $\beta = 1/T$ ,  $T > 0$ -temperature,  $Z_n^{-1}$  is the normalizing factor,  $\{\tilde{h}_x = (\tilde{h}_{1,x}, \dots, \tilde{h}_{q,x}) \in \mathbb{R}^q, x \in V\}$  is a collection of vectors and  $H_n(\sigma_n)$  is the restriction of Hamiltonian on  $V_n$ .

We say that the probability distributions (2.2) are compatible if for all  $n \geq 1$  and  $\sigma_{n-1} \in \Phi^{V_{n-1}}$ :

$$\sum_{\omega_n \in \Phi^{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) = \mu_{n-1}(\sigma_{n-1}). \quad (2.3)$$

Here  $\sigma_{n-1} \vee \omega_n$  is the concatenation of the configurations. In this case, there exists a unique measure  $\mu$  on  $\Phi^V$  such that, for all  $n$  and  $\sigma_n \in \Phi^{V_n}$ ,

$$\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu_n(\sigma_n).$$

Such a measure is called a *splitting Gibbs measure* (SGM) corresponding to the Hamiltonian (2.1) and the vector-valued function  $\tilde{h}_x, x \in V$ .

The following statement describes conditions on  $\tilde{h}_x$  guaranteeing compatibility of  $\mu_n(\sigma_n)$ .

**Theorem 1.** (see [3], [18, p.106]) *The probability distributions  $\mu_n(\sigma_n)$ ,  $n = 1, 2, \dots$ , in (2.2) are compatible iff for any  $x \in V \setminus \{x^0\}$  the following equation holds:*

$$h_x = \sum_{y \in S(x)} F(h_y, \theta), \quad (2.4)$$

where  $F : h = (h_1, \dots, h_{q-1}) \in \mathbb{R}^{q-1} \rightarrow F(h, \theta) = (F_1, \dots, F_{q-1}) \in \mathbb{R}^{q-1}$  is defined as

$$F_i = \ln \left( \frac{(\theta - 1)e^{h_i} + \sum_{j=1}^{q-1} e^{h_j} + 1}{\theta + \sum_{j=1}^{q-1} e^{h_j}} \right),$$

$\theta = \exp(J\beta)$ ,  $S(x)$  is the set of direct successors of  $x$  and  $h_x = (h_{1,x}, \dots, h_{q-1,x})$  with

$$h_{i,x} = \tilde{h}_{i,x} - \tilde{h}_{q,x}, \quad i = 1, \dots, q-1. \quad (2.5)$$

From Theorem 1 it follows that for any  $h = \{h_x, x \in V\}$  satisfying (2.4) there exists a unique SGM  $\mu$  for Potts model.

To compare with the literature we remark that the quantities  $\exp(\tilde{h}_{i,x})$  define a boundary law in the sense of Definition 12.10 in Georgii's book [5]. Compare also Theorem 12.2 therein which describes the connection between boundary laws and finite-volume marginals of splitting Gibbs measures for general spin models and general volumes (of which formula (2.2) is a special case). Looking to marginals on volumes which consist of two adjacent sites from that expression in particular the relation between boundary law and the transition matrix for the associated Markov chain (splitting Gibbs measure) is immediately obtained.

*Translation-invariant SGMs.* A translation-invariant splitting Gibbs measure (TISGM) corresponds to a solution  $h_x$  of (2.4) with  $h_x = h = (h_1, \dots, h_{q-1}) \in R^{q-1}$  for all  $x \in V$ . Then from equation (2.4) we get  $h = kF(h, \theta)$ , and denoting  $z_i = \exp(h_i)$ ,  $i = 1, \dots, q-1$ , the last equation can be written as

$$z_i = \left( \frac{(\theta - 1)z_i + \sum_{j=1}^{q-1} z_j + 1}{\theta + \sum_{j=1}^{q-1} z_j} \right)^k, \quad i = 1, \dots, q-1. \quad (2.6)$$

In [10] all solutions of the equation (2.6) are given. By these solutions the full set of TISGMs is described. In particular, the following results are obtained which will be the starting point of the present analysis and which we repeat for convenience of the reader.

**Theorem 2.** [10] *For any solution  $z = (z_1, \dots, z_{q-1})$  of the system of equations (2.6) there exist  $M \subset \{1, \dots, q-1\}$  and  $z^* > 0$  such that*

$$z_i = \begin{cases} 1, & \text{if } i \notin M \\ z^*, & \text{if } i \in M. \end{cases}$$

Thus any TISGM of the Potts model corresponds to a solution of the following equation

$$z = f_m(z) \equiv \left( \frac{(\theta + m - 1)z + q - m}{mz + q - m - 1 + \theta} \right)^k, \quad (2.7)$$

for some  $m = 1, \dots, q-1$ .

Denote

$$\theta_m = 1 + 2\sqrt{m(q-m)}, \quad m = 1, \dots, q-1. \quad (2.8)$$

It is easy to see that

$$\theta_m = \theta_{q-m} \quad \text{and} \quad \theta_1 < \theta_2 < \dots < \theta_{[\frac{q}{2}]-1} < \theta_{[\frac{q}{2}]} \leq q+1. \quad (2.9)$$

**Proposition 1.** [10] *Let  $k = 2$ ,  $J > 0$ .*

1. If  $\theta < \theta_1$  then there exists a unique TISGM;
2. If  $\theta_m < \theta < \theta_{m+1}$  for some  $m = 1, \dots, [\frac{q}{2}] - 1$  then there are  $1 + 2 \sum_{s=1}^m \binom{q}{s}$  TISGMs which correspond to the solutions  $z_i \equiv z_i(\theta, q, m) = x_i^2(m, \theta)$ ,  $i = 1, 2$  of (2.7), where

$$x_{1,2}(m, \theta) = \frac{\theta - 1 \pm \sqrt{(\theta - 1)^2 - 4m(q - m)}}{2m}. \quad (2.10)$$

3. If  $\theta_{[\frac{q}{2}]} < \theta \neq q + 1$  then there are  $2^q - 1$  TISGMs;
4. If  $\theta = q + 1$  the number of TISGMs is as follows

$$\begin{cases} 2^{q-1}, & \text{if } q - \text{odd} \\ 2^{q-1} - \binom{q-1}{q/2}, & \text{if } q - \text{even} \end{cases}$$

5. If  $\theta = \theta_m$ ,  $m = 1, \dots, [\frac{q}{2}]$ ,  $(\theta_{[\frac{q}{2}]} \neq q + 1)$  then number of TISGMs is

$$1 + \binom{q}{m} + 2 \sum_{s=1}^{m-1} \binom{q}{s}.$$

### 3. FUZZY TRANSFORMATION TO ISING MODEL WITH AN EXTERNAL FIELD

In order to study extremality of a TISGM corresponding to a solution  $z > 0$  of (2.7), (i.e. to a vector  $l = (l_1, \dots, l_q) \in R^q$  with  $m$  coordinates equal to  $z$  and  $q - m$  coordinates equal to 1) we divide the coordinates of this vector to two classes: We write  $l'$  and  $l''$ , where  $l'$  has  $m$  coordinates each equal to  $z$  and  $l''$  has  $q - m$  coordinates each equal to 1.

Without loss of generality, by relabeling of coordinates, we can take  $l$ ,  $l'$  and  $l''$  as follows:

$$l = (\underbrace{z, z, \dots, z}_m, \underbrace{1, 1, \dots, 1}_{q-m}) = ((\underbrace{z, z, \dots, z}_m), (\underbrace{1, 1, \dots, 1}_{q-m})) = (l', l'').$$

Define a (fuzzy) map  $T : \{1, 2, \dots, q\} \rightarrow \{1', 2'\}$  as

$$T(i) = \begin{cases} 1', & \text{if } i \leq m \\ 2', & \text{if } i \geq m + 1. \end{cases} \quad (3.1)$$

This map identifies spin-values which have the same value of the boundary law and are treated in an equal fashion by the transition matrix. We extend this map to act on infinite-volume spin configurations and measures in the infinite volume.

We note that a TISGM corresponding to a vector  $l \in R^q$  is a tree-indexed Markov chain with states  $\{1, 2, \dots, q\}$  and transition probabilities matrix  $\mathbb{P} = (P_{ij})$  with

$$P_{ij} = \frac{l_j \exp(J\beta\delta_{ij})}{\sum_{r=1}^q l_r \exp(J\beta\delta_{ir})}. \quad (3.2)$$

From (3.2) we get

$$P_{ij} = \begin{cases} \theta z / Z_1, & \text{if } i = j, i \in \{1, \dots, m\} \\ z / Z_1, & \text{if } i \neq j, i, j \in \{1, \dots, m\} \\ 1 / Z_1, & \text{if } i \in \{1, \dots, m\}, j \in \{m+1, \dots, q\} \\ z / Z_2, & \text{if } i \in \{m+1, \dots, q\}, j \in \{1, \dots, m\} \\ \theta / Z_2, & \text{if } i = j, i \in \{m+1, \dots, q\} \\ 1 / Z_2, & \text{if } i \neq j, i, j \in \{m+1, \dots, q\}, \end{cases} \quad (3.3)$$

where

$$Z_1 = (\theta + m - 1)z + q - m, \quad Z_2 = mz + \theta + q - m - 1.$$

The fuzzy map  $T$  reduces the matrix  $\mathbb{P}$  to the  $2 \times 2$  matrix  $T(\mathbb{P}) = \hat{\mathbb{P}} = (p_{ij})_{i,j=1',2'}$  with

$$p_{ij} = \begin{cases} \frac{(\theta+m-1)z}{Z_1}, & \text{if } i = j = 1' \\ \frac{q-m}{Z_1}, & \text{if } i = 1', j = 2' \\ \frac{mz}{Z_2}, & \text{if } i = 2', j = 1' \\ \frac{\theta+q-m-1}{Z_2}, & \text{if } i = j = 2'. \end{cases} \quad (3.4)$$

More precisely this means: Consider the translation-invariant tree-indexed Markov chain  $\mu$  with transition matrix given by (3.3). Then its image measure  $T(\mu)$  under the site-wise application of the fuzzy map  $T$  is a tree-indexed Markov chain with local state space  $1', 2'$  with the transition matrix given by (3.4).

Note that an application of a transformation to an initial Gibbs measure which is not adapted to the structure of the transition matrix of the initial chain (for example a fuzzy map which identifies spin values which have different values of the boundary law) would in general not give rise to a Markov chain (and possibly not even to a Gibbs measure with an absolutely summable interaction potential).

We also observe the following fact: If an initial Markov chain  $\mu$  is extremal in the set of Gibbs-measures for the Potts model it implies its triviality on the tail-sigma algebra of the  $q$ -spin events. But this implies that the mapped chain  $T(\mu)$  is trivial on its own tail-sigma algebra (since the latter can be identified with the sub-sigma algebra of event in the tail-sigma algebra of the  $q$ -spin events which do not distinguish spin-values which have the same values of the fuzzy variable).

We note that the matrix  $\hat{\mathbb{P}}$  has the two eigenvalues 1 and

$$\lambda_2(\hat{\mathbb{P}}) = \frac{(\theta + m - 1)z}{Z_1} + \frac{\theta + q - m - 1}{Z_2} - 1. \quad (3.5)$$

Let us comment on some general properties of transition matrices which have a two-block symmetry as our transition matrix has: Our matrix  $\mathbb{P}$  is a stochastic  $q \times q$ -matrix, in block-form which has as parameters the block size  $m$  and can be written in terms of four real independent parameters  $p_1, p_2, q_1, q_2$ .

Here  $p_1$  is the transition rate for going from a state in  $1'$  to a different state in  $1'$ ,  
 $q_1$  is the transition rate for going from a state in  $2'$  to a different state in  $2'$ ,  
 $p_2$  is the transition rate for going from a state in  $1'$  to a state in  $2'$ ,  
 $q_2$  is the transition rate for going from a state in  $2'$  to a state in  $1'$ .  
The form is

$$\mathbb{P} = (\tilde{\mathbb{P}}_{i,j})_{i,j=1,\dots,q} = \begin{pmatrix} aE_m & 0 \\ 0 & bE_{q-m} \end{pmatrix} + \begin{pmatrix} p_1 I_{m,m} & p_2 I_{m,q-m} \\ q_2 I_{q-m,m} & q_1 I_{q-m,q-m} \end{pmatrix} \quad (3.6)$$

where  $I_{k,l}$  is the  $k \times l$  matrix which has all matrix elements equal to 1 and  $E_k$  is the  $k$ -dimensional unity matrix. Here necessarily  $a = 1 - mp_1 - (q - m)p_2$  and  $b = 1 - mq_2 - (q - m)q_1$ , so that the matrix is stochastic.

We consider the action of the fuzzy map on probability vectors and introduce the linear map from  $\mathbb{R}^q$  to  $\mathbb{R}^2$  given by

$$Lv = \begin{pmatrix} \sum_{i=1}^m v_i \\ \sum_{i=m+1}^q v_i \end{pmatrix}.$$

In matrix form we have  $L = \begin{pmatrix} I_{1,m} & 0 \\ 0 & I_{1,q-m} \end{pmatrix}$ .

The following lemma is obvious.

**Lemma 1.** *We have*

$$L\mathbb{P} = \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} mp_1 & mp_2 \\ (q-m)q_2 & (q-m)q_1 \end{pmatrix} \right) L =: \mathbb{P}'L.$$

We also have

$$L\mathbb{P}^t = \left( \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} mp_1 & mq_2 \\ (q-m)p_2 & (q-m)q_1 \end{pmatrix} \right) L = \hat{\mathbb{P}}^t L,$$

where  $\hat{\mathbb{P}}$  (see (3.4)) is the stochastic matrix, namely the transition matrix for the coarse-grained chain with two states  $1', 2'$ .

**Proposition 2.** *The matrix  $\mathbb{P}^t$  is diagonalisable (from the right) and has the eigenvalues  $1, \lambda_2(\hat{\mathbb{P}}), a, b$ . The dimension of the eigenspace to  $a$  is  $m - 1$ . The dimension of the eigenspace to  $b$  is  $q - m - 1$ .*

*Proof.* Suppose  $v$  is a right-eigenvector of  $\mathbb{P}^t$  for the eigenvalue  $\lambda$ , so  $\mathbb{P}^t v = \lambda v$ . Then  $\hat{\mathbb{P}}^t Lv = L\mathbb{P}^t v = \lambda Lv$ . Two cases are possible:

Case 1:  $Lv \neq 0$  and hence  $\lambda$  is an eigenvalue for  $\hat{\mathbb{P}}$ , too. The two eigenvalues for the two-by-two matrix can be easily evaluated, one eigenvalue is equal to 1, call the other one  $\lambda_2(\hat{\mathbb{P}})$ .

Case 2:  $Lv = 0$ . Then we must have

$$\begin{aligned} \mathbb{P}^t v &\equiv \mathbb{P}^t \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} aE_m & 0 \\ 0 & bE_{q-m} \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \\ &= \lambda \begin{pmatrix} u \\ w \end{pmatrix} \end{aligned} \quad (3.7)$$



In order to have an eigenvalue  $v$  at least one of the components  $u$  or  $w$  has to be nonzero.

Hence the possible eigenvalues are  $a, b$  (which are allowed to be equal or not).

The eigenvectors corresponding to  $a$  are of the form  $\begin{pmatrix} u \\ 0 \end{pmatrix}$  with  $\sum_{i=1}^m u_i = 0$ . We have  $m - 1$  linearly independent of them.

The eigenvectors corresponding to  $b$  are of the form  $\begin{pmatrix} 0 \\ w \end{pmatrix}$  with  $\sum_{i=m+1}^q w_i = 0$ . Clearly we have  $q - m - 1$  of them.  $\square$

**Corollary 1.** *The matrix  $\mathbb{P}$  defined by (3.3) with  $m \leq [q/2]$  has the following eigenvalues*

$$\begin{aligned} \{1, b, \lambda_2(\hat{\mathbb{P}})\}, \quad & \text{if } m = 1 \\ \{1, a, b, \lambda_2(\hat{\mathbb{P}})\}, \quad & \text{if } m \geq 2, \end{aligned} \quad (3.8)$$

where

$$a = \frac{(\theta - 1)z}{Z_1}, \quad b = \frac{(\theta - 1)\sqrt[k]{z}}{Z_1}, \quad \lambda_2(\hat{\mathbb{P}}) = \frac{[\theta - 1 + (1 - \sqrt[k]{z})m]z}{Z_1}. \quad (3.9)$$

*Proof.* Since  $z$  is a solution to (2.7) we have  $Z_1 = \sqrt[k]{z}Z_2$ . In case  $m = 1$  we have  $p_1 = 0$  and the above mentioned condition  $Lu = 0$  gives  $u = u_1 = 0$ , i.e.  $a$  can not be an eigenvalue. For  $m \geq 2$  using  $a + p_1 = \frac{\theta z}{Z_1}$ ,  $p_1 = \frac{z}{Z_1}$ ,  $b + q_1 = \frac{\theta}{Z_2}$ ,  $q_1 = \frac{1}{Z_2}$  and (3.5) we get (3.8).  $\square$

For  $a, b, \lambda_2$  given by (3.8) denote

$$\hat{\lambda} = \hat{\lambda}(k, q, m, \theta) = \max\{a, b, |\lambda_2(\hat{\mathbb{P}})|\}.$$

For  $k = 2$ , using formula (2.10) of  $z \in \{x_1^2, x_2^2\}$ , for  $m = 1$  we obtain  $\hat{\lambda} < b$  and for  $m \geq 2$  we have

$$|\lambda_2(\hat{\mathbb{P}})| < \hat{\lambda} = \begin{cases} b, & \text{if } z < 1 \\ a, & \text{if } z > 1. \end{cases} \quad (3.10)$$

It turns out that we can lift the action of the coarse graining to obtain an effective Hamiltonian for the coarse-grained Markov chain which has the interpretation of an Ising model in a magnetic field which is non-vanishing if and only if  $2m = q$ . Note that this Hamiltonian is not just the logarithm of the transition matrix of the Potts chain (which would be  $z$ -dependent) but it will be  $z$ -independent. More precisely we have the following proposition.

**Proposition 3.** *Let  $k \geq 2$  be an integer. Consider a translation invariant Markov chain  $\mu_{\theta, m, z}$  for the Potts model with coupling constant parameterized  $\theta$  of the type we constructed above, given by  $m \leq q/2$  and a corresponding choice of the (up to three) values of the boundary law  $z$ .*

Then there exist a coupling constant  $J' = J'(\theta, m)$  and a magnetic field value  $h' = h'(\theta, m)$ , where both values are independent of the boundary law  $z$ , such that the coarse-grained measure  $T(\mu_{\theta, m, z})$  is a Gibbs measure for the Hamiltonian

$$H'(\phi) = -J' \sum_{\langle x, y \rangle} \phi(x)\phi(y) - h' \sum_x \phi(x) \quad (3.11)$$

of the corresponding Ising model with spin variables  $\phi(x) = \pm 1$  on the Cayley tree of order  $k$  which is a tree-indexed Markov chain (splitting Gibbs measure) which has the given matrix  $M$  as its transition matrix. Here we have identified the fuzzy class 1' with the Ising spin value +1 and the fuzzy class 2' with the Ising spin value -1.

For the coupling constant we have

$$e^{4J'} = \frac{(\theta + m - 1)(\theta + q - m - 1)}{(q - m)m} \quad (3.12)$$

For the magnetic field we have

$$e^{\frac{4h'}{k+1}} = \frac{(\theta + m - 1)m}{(q - m)(\theta + q - m - 1)} \left(\frac{z}{s}\right)^2 \quad (3.13)$$

where  $z$  denotes the value of the boundary law for the Potts model and  $(s, 1)$  denotes the value of the corresponding boundary law for the Ising model.

The boundary laws for the Potts model and the corresponding Ising model satisfy the relation

$$s = \left(\frac{m}{q - m}\right)^{\frac{k}{k+1}} z \quad (3.14)$$

which is independent of the choice of the solution at fixed  $m$  and makes the magnetic field  $h'$  independent of the choice of  $z$  at fixed  $m$ .

It is interesting to compare this result with Proposition 4.1 in [9] which states that any fuzzy image of the open b.c. condition Potts measure (which is obtained for  $z = 1$ ) is quasilocal. Here we extent this result to the larger class of Markov chains with fixed  $m$  and remark that the Hamiltonian of the fuzzy model with classes  $m$  and  $q - m$  stays the same when we take a different boundary law  $z$ . Hence it suffices to look at the free measure to construct this Hamiltonian.

*Proof.* The transition matrix of the Ising model whose Hamiltonian is to be constructed has the form

$$\hat{Q} = \left( e^{J'\phi(x)\phi(y) + \frac{h'}{k+1}\phi(x) + \frac{h'}{k+1}\phi(y)} \right)_{\phi(x)=\pm 1, \phi(y)=\pm 1} = \begin{pmatrix} e^{J' + \frac{2h'}{k+1}} & e^{-J'} \\ e^{-J'} & e^{J' - \frac{2h'}{k+1}} \end{pmatrix} \quad (3.15)$$

(compare e.g. (12.20) of Georgii's book). The corresponding equation for a boundary law  $(s_x, 1)$  (which we allow to be  $x$ -dependent at this stage) for the Ising model

is written as

$$s_x = \prod_{y \in S(x)} \frac{e^{J' + \frac{2h'}{k+1}} s_y + e^{-J'}}{e^{-J'} s_y + e^{J' - \frac{2h'}{k+1}}}. \quad (3.16)$$

Suppose we have a homogeneous solution  $s$  of this equation for the boundary law of the Ising model. Then the corresponding transition matrix is

$$\hat{P}_s = \begin{pmatrix} \frac{e^{J' + \frac{2h'}{k+1}} s}{e^{J' + \frac{2h'}{k+1}} s + e^{-J'}} & \frac{e^{-J'}}{e^{J' + \frac{2h'}{k+1}} s + e^{-J'}} \\ \frac{e^{-J'} s}{e^{-J'} s + e^{J' - \frac{2h'}{k+1}}} & \frac{e^{J' - \frac{2h'}{k+1}}}{e^{-J'} s + e^{J' - \frac{2h'}{k+1}}} \end{pmatrix}. \quad (3.17)$$

Recall the form of  $M = \hat{P}$  by which we denote the stochastic  $2 \times 2$ -transition matrix we obtained from the application of the fuzzy map  $T$  to an  $m$ - and  $z$ -dependent splitting Gibbs measure  $\mu$  for the Potts model as described above.

Equating this transition matrix with the transition matrix obtained from the coarse-graining of the boundary laws for the Potts model for given  $m$  and  $z$  we find

$$e^{2J' + \frac{2h'}{k+1}} s = \frac{p_{11}}{p_{12}} = \frac{\theta + m - 1}{q - m} \quad (3.18)$$

and

$$e^{-2J' + \frac{2h'}{k+1}} s = \frac{p_{21}}{p_{22}} = \frac{m}{\theta + q - m - 1} z. \quad (3.19)$$

Taking quotient (respectively product) of these equations the formulae for coupling and magnetic field (3.12) (3.13) follow. To see the relation (3.14) between homogeneous boundary laws of the Potts model and the Ising model we start with the homogenous version of the Ising boundary law equation (3.16) to obtain

$$\begin{aligned} s &= s^{-k} \left( \frac{e^{2J' + \frac{2h'}{k+1}} s + 1}{1 + (s e^{-2J' + \frac{2h'}{k+1}})^{-1}} \right)^k \\ &= \left( \frac{m}{q - m} \frac{z}{s} \right)^k \left( \frac{(\theta + m - 1)z + q - m}{mz + q - m - 1 + \theta} \right)^k \\ &= \left( \frac{m}{q - m} \frac{z}{s} \right)^k z, \end{aligned} \quad (3.20)$$

where we have substituted in the second line (3.18) and (3.19) to bring the Potts parameters into play and recognized the r.h.s. of the equation for the boundary law of the Potts model  $z = f_m(z)$  to get the third line. This proves the relation between the boundary laws (3.14) independently of the choice of the solution for  $m$  fixed (and independently of temperature of the initial Potts model).

Let us go back to (3.16) which is the functional (spatially dependent) equation for the boundary law for the Ising model at fixed  $J', h'$ . The relation between boundary laws also works for spatially dependent boundary laws and we get that  $s_x$  is a solution for the Ising model (3.16) if and only if the quantity  $z_x := s_x \left( \frac{q-m}{m} \right)^{\frac{k}{k+1}}$  is a solution of the functional equation

$$z_x = \prod_{y \in S(x)} \frac{(\theta + m - 1)z_y + q - m}{mz_y + q - m - 1 + \theta}. \quad (3.21)$$

The proof of this statement uses the same substitutions as in the homogeneous case.  $\square$

As it was mentioned above for each fixed  $m$ , the equation (2.7) has up to three solutions:  $z_0 = 1, z_i = z_i(\theta, q, m), i = 1, 2$ . Denote by  $\mu_i = \mu_i(\theta, m)$  the TISGM of the Potts model which corresponds to the solution  $z_i$  and by  $T(\mu_i)$  its image for the Ising model (3.11).

Define

$$g(z) = \sqrt[k]{f_m(z)} = \frac{(\theta + m - 1)z + q - m}{mz + q - m - 1 + \theta}.$$

By simple analysis we get the following

- Lemma 2.**
- i. For  $\theta > 1$  the function  $g(z)$ ,  $z > 0$  has the following properties:
    - a)  $\{z : g^k(z) = f_m(z) = z\} = \{1, z_1, z_2\}$ ;
    - b)  $a < g(z) < A$ , with  $a = \frac{q-m}{q+\theta-m-1}$ ,  $A = \frac{\theta+m-1}{m}$ ;
    - c)  $\frac{d}{dz}g(z) = \frac{(\theta-1)(\theta+q-1)}{(mz+\theta+q-m-1)^2} > 0$ ;
    - d)  $\frac{d^2}{dz^2}g(z) < 0$ ,  $z > 0$ .
  - ii. If  $m \leq q/2$  then for solutions  $z_1$  and  $z_2$  mentioned in Proposition 1 the following statements hold

$$\begin{aligned} 1 < z_1 = z_2, & \text{ if } \theta = \theta_m \\ 1 < z_1 < z_2, & \text{ if } \theta_m < \theta < \theta_c = q + 1 \\ 1 = z_1 < z_2, & \text{ if } \theta = q + 1 \\ z_1 < 1 < z_2, & \text{ if } q + 1 < \theta. \end{aligned}$$

In Figure 1 the functions  $z_i = x_i^2(m, \theta)$ ,  $i = 1, 2$  are shown for  $q = 8$ ,  $m = 1, 2, 3$ .

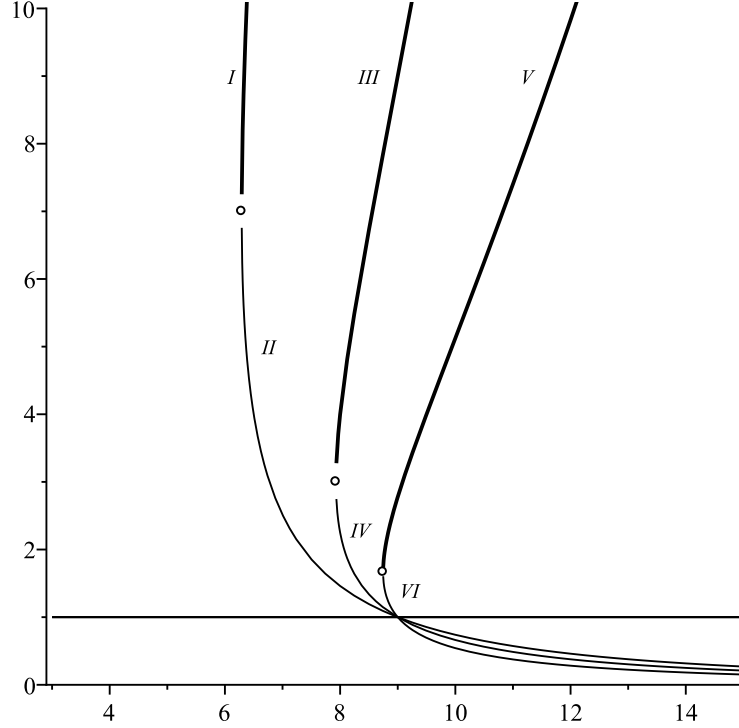


Fig. 1. The graphs of functions  $z_i = z_i(m, \theta)$ ,  $i = 1, 2$ , for  $q = 8$  and  $m = 1, 2, 3$ . Circle dots having coordinates  $(\theta_m, \frac{q-m}{m})$ ,  $m = 1, 2, 3$  separate graph of  $z_1$  from graph of  $z_2$ . The graph of  $z_2(1, \theta)$  is the curve *I*. The graph of  $z_1(1, \theta)$  is the curve *II*.  $z_2(2, \theta)$  is *III*.  $z_1(2, \theta)$  is *IV*,  $z_2(3, \theta)$  is *V*.  $z_1(3, \theta)$  is *VI*. The intersection point of  $z_1$  functions has coordinate  $(\theta_c, 1) = (9, 1)$ .

It is known (see, e.g., [5]) that for all  $\beta > 0$ , the Gibbs measures form a non-empty convex compact set in the space of probability measures. Extreme measures, i.e., extreme points of this set are associated with pure phases. Furthermore, any Gibbs measure is an integral of extreme ones (the extreme decomposition).

Write  $\varphi \leq \varphi'$  if configurations  $\varphi$  and  $\varphi'$  obey  $\varphi(x) \leq \varphi'(x)$  for all  $x \in V$ . This partial order defines a concept of a monotone increasing and monotone decreasing function  $f : \{1, 2\}^V \rightarrow R$ . For two probability measures  $\mu_1$  and  $\mu_2$  we then write  $\mu_1 \leq \mu_2$  if  $\int f d\mu_1 \leq \int f d\mu_2$  for each monotone increasing  $f$ . It turns out that for the ‘extreme’ configurations,  $\varphi_i$  with  $\varphi_i(x) \equiv i$ ,  $i = 1, 2$ , there exist the limiting Gibbs measures  $\nu_i$  with boundary configuration  $\varphi_i$  (both measure sequences are monotone).  $\nu_1, \nu_2$  are TIGMs and possess the following minimality and maximality properties:  $\nu_1 \leq \mu \leq \nu_2$  for all Gibbs measure  $\mu$ . Because of that, they are both extreme. The question of whether a Gibbs measure is non-unique is then reduced to whether  $\nu_1 = \nu_2$ .

In the language of the boundary laws (see [5]) the maximal and minimal measures correspond to maximal and minimal solutions of (2.7). More precisely, we have the following

**Proposition 4.** 1. *Each SGM of the model (3.11) corresponds to a solution  $z_x \in R$ ,  $x \in V$  of the following functional equation*

$$z_x = \prod_{y \in S(x)} \frac{(\theta + m - 1)z_y + q - m}{mz_y + q - m - 1 + \theta}. \quad (3.22)$$

2. *If  $z_x$  is a solution of (3.22) then*

$$\min\{1, z_1, z_2\} \leq z_x \leq \max\{1, z_1, z_2\}, \quad (3.23)$$

*where  $z_1, z_2$  are solutions to (2.7).*

*Proof.* 1. This was proved already below (3.20).

2. For  $z > 0$  by Lemma 2 we have  $a < g(z) < A$ . Using this inequality from (3.22) we get

$$a^k < z_x < A^k.$$

Now we consider the function  $g(z)$  on  $[a^k, A^k]$  and on this segment, using Lemma 2, we get the estimations  $g(a^k) < g(z) < g(A^k)$ . Then

$$(g(a^k))^k < z_x < (g(A^k))^k.$$

Iterating this argument we obtain

$$f_m^n(a) < z_x < f_m^n(A),$$

where  $f_m(z)$  is defined in (2.7), and  $f_m^n$  is its  $n$ th iteration. It is easy to see that  $\max\{1, z_1, z_2\} \leq f_m^n(A)$  and the sequence  $f_m^n(A)$  monotone decreasing. Thus the sequence has a limit  $\alpha$ , with  $\alpha \geq \max\{1, z_1, z_2\}$ . This limit point must be a fixed point for  $f_m$ . But since the function  $f_m$  has no fixed point in  $(\max\{1, z_1, z_2\}, +\infty)$  we get that  $\alpha = \max\{1, z_1, z_2\}$ .  $\square$

Let  $\mathcal{G}(H')$  the set of all SGMs of the model (3.11).

**Theorem 3.** *For the model (3.11) on the Cayley tree of order  $k \geq 2$  the following statements are true*

- (1) *If  $\theta < \theta_m$  then there is unique TISGM  $T(\mu_0)$ .*
- (2) *If  $\theta = \theta_m$  or  $\theta = \theta_c = \frac{q+k-1}{k-1}$  then there are 2 TISGMs  $T(\mu_0), T(\mu_1)$ .*
- (3) *If  $\theta_m < \theta \neq \theta_c$  then there are 3 TISGMs  $T(\mu_0), T(\mu_1), T(\mu_2)$ . Moreover, at least two of these measures are extreme in  $\mathcal{G}(H')$ .*

*Proof.* The measures  $T(\mu_i)$ ,  $i = 0, 1, 2$  correspond to solutions  $z_0 = 1$ ,  $z_1$  and  $z_2$  of (2.7). The extremality of two of these measures can be deduced using the minimality and maximality of the corresponding values of solutions. Assume that  $z_2$  is the maximal solution and  $T(\mu_2)$  is non-extreme, i.e., is decomposed:

$$T(\mu_2) = \int T(\mu)(\omega) \nu(d\omega).$$

Then for any vertex  $x \in V$  we have

$$z_2 = \int z_x(\omega) \nu(d\omega). \quad (3.24)$$

By Proposition 4  $z_2$  is an extreme point in the set of all solutions  $z_x$ , (3.24) holds if  $z_x(\omega) = z_2$  for almost all  $\omega$ . Hence,  $T(\mu_2)$  is extreme.  $\square$

We recall that non-extremality of a coarse-grained measure implies non-extremality for the corresponding measure for the initial Potts model. In the opposite direction there is no direct implication available.

#### 4. CONDITIONS FOR NON-EXTREMALITY

Let us continue with the investigation of the coarse-grained chains  $T(\mu)$  with a focus on criteria for non-extremality which are based on properties of the corresponding  $2 \times 2$ -transition matrix as it depends on coupling strength of the initial Potts model, the block-size  $m$  and the choice of a branch of the boundary law.

In case when there are three solutions, we denote the middle solution by

$$z_{\text{mid}} = \{1, z_1, z_2\} \setminus \{\min\{1, z_1, z_2\}, \max\{1, z_1, z_2\}\}.$$

Note that for  $2m < q$  we have  $z_{\text{mid}} = \max\{1, z_1\}$ .

Let  $T(\mu_{\text{mid}})$  be the TISGM which corresponds to  $z_{\text{mid}}$ . Recall that  $\mu_i$  is the TISGM of the Potts model which corresponds to the boundary law  $z_i(m)$ , for the branches  $i = 1, 2$ .

Define the following numbers:

$$\begin{aligned} \theta_0 &= 1 + q + 2\sqrt{2m(q-m)}, \quad \hat{\theta}_0 = 1 + (\sqrt{2} + 1)q, \\ \hat{\theta} &= (\sqrt{2} - 1)q + 2m + 1, \quad \theta^* = 1 + (\sqrt{2} + 1)q - 2m. \end{aligned} \quad (4.1)$$

**Theorem 4.** *Let  $k = 2$ ,  $2m < q$ . Then the following statements hold.*

(i) *If  $\theta \geq q + 1$  then  $T(\mu_{\text{mid}}) = T(\mu_0)$  and*

$$T(\mu_0) \text{ is } \begin{cases} \text{extreme,} & \text{if } \theta \leq \theta_0 \\ \text{non-extreme,} & \text{if } \theta > \hat{\theta}_0; \end{cases}$$

(ii) *Assume one of the following conditions is satisfied:*

- a)  $2 \leq m \leq q/7$  and  $\theta \in [\theta_m, \hat{\theta})$ ;
- b)  $\theta \in (\theta^*, +\infty)$ .

*Then  $\mu_1(\theta, m)$  is non-extreme.*

(iii) *Assume one of the following conditions is satisfied:*

- c)  $2 \leq m \leq q/7$  and  $\theta \geq \theta_m$ ;
- d)  $q < 7m$ ,  $m \geq 2$  and  $\theta \in (\hat{\theta}, +\infty)$ .

*Then  $\mu_2(\theta, m)$  is non-extreme. (See Fig.2-4)*

- (iv) If  $\theta_m \leq \theta < q + 1$  then  $T(\mu_{\text{mid}}) = T(\mu_1)$ . Moreover, if  $q \geq 85$  then for each  $m \leq \lfloor \frac{q}{85} \rfloor$  there are two critical values  $\bar{\theta}, \bar{\bar{\theta}} \in (\theta_m, q + 1)$ , with  $\bar{\theta} < \bar{\bar{\theta}}$  such that  $T(\mu_1)$  is non-extreme if  $\theta \in (\bar{\theta}, \bar{\bar{\theta}})$ . Moreover,  $\bar{\theta}, \bar{\bar{\theta}}$  are solutions to

$$(\theta - 1)^3 - (\sqrt{2} - 1)q(\theta - 1)^2 - 2(2\sqrt{2} - 1)m(q - m)(\theta - 1) + 2qm(q - m) = 0. \quad (4.2)$$

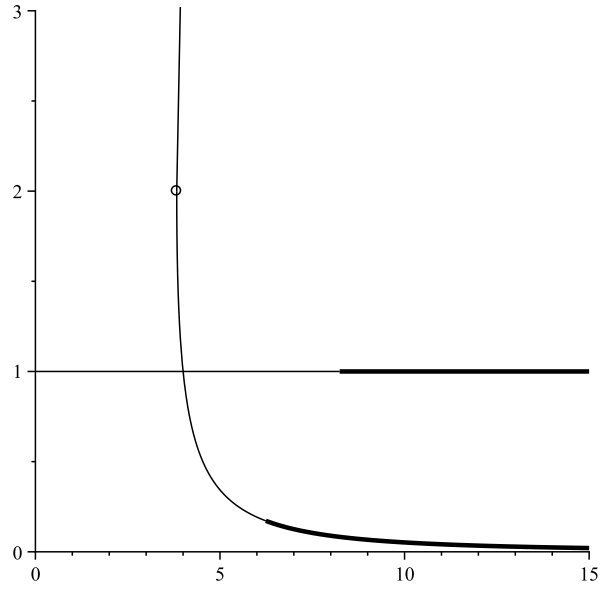


Fig. 2. The graphs of functions  $z_i = z_i(m, \theta)$ ,  $i = 1, 2$ , for  $q = 3$  and  $m = 1$ . The circle dot having coordinate  $(\theta_1, 2)$  separates graph of  $z_1$  from graph of  $z_2$ . The bold curves correspond to regions of solutions where corresponding TISGM is non-extreme.



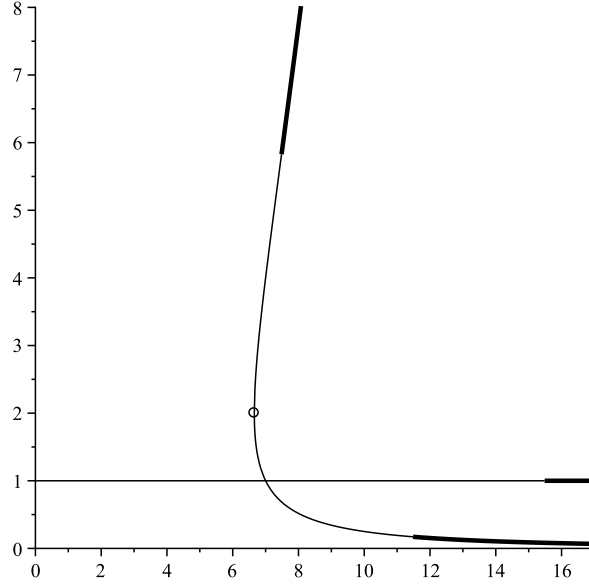


Fig. 3. The graphs of functions  $z_i = z_i(m, \theta)$ ,  $i = 1, 2$ , for  $q = 6$  and  $m = 2$ . The circle dot having coordinate  $(\theta_2, 2)$  separates graph of  $z_1$  from graph of  $z_2$ . The bold lines correspond to regions of solutions where corresponding TISGM is non-extreme.

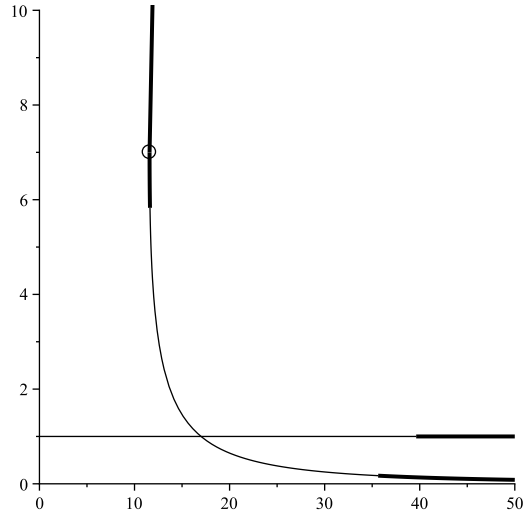


Fig. 4. The graphs of functions  $z_i = z_i(m, \theta)$ ,  $i = 1, 2$ , for  $q = 16$  and  $m = 2$ . The circle dot having coordinate  $(\theta_2, 7)$  separates graph of  $z_1$  from graph of  $z_2$ . The bold lines correspond to regions of solutions where corresponding TISGM is non-extreme.

*Proof.* The equality  $T(\mu_{\text{mid}}) = T(\mu_0)$  follows from the second part of Lemma 2. To check the extremality we apply arguments of a reconstruction on trees [2], [8], [12], [15], [16]. Consider Markov chains with states  $\{1', 2'\}$  and transition probabilities  $p_{ij}$  defined by (3.4). It is known that a sufficient condition for non-extremality (which is equivalent to solvability of the associated reconstruction) of a Gibbs measure  $T(\mu)$  corresponding to the matrix  $T(\mathbb{P})$  is that  $k\lambda_2^2 > 1$ , where  $\lambda_2$  is the second largest (in absolute value) eigenvalue of  $T(\mathbb{P})$  [8]. On the other hand, Martin in [3] gives the following condition for extremality (non-reconstructibility)

$$k \left( \sqrt{T(P_{11})T(P_{22})} - \sqrt{T(P_{12})T(P_{21})} \right)^2 \leq 1.$$

(i) In case  $z = 1$  the matrix  $T(\mathbb{P})$  is

$$T(\mathbb{P}) = \frac{1}{\theta + q - 1} \begin{pmatrix} \theta + m - 1 & q - m \\ m & \theta + q - m - 1 \end{pmatrix}. \quad (4.3)$$

Simple calculations show that above-mentioned conditions of extremality for  $T(\mu_0)$  (i.e. for the matrix (4.3)) are equivalent to the conditions on  $\theta$  as mentioned in the-orem.

(ii) **Case**  $m \geq 2$ .

*Subcase:*  $\theta_m \leq \theta < q + 1$ . In this case by Lemma 2 we have  $z_1 > 1$ . By (3.10) to prove non-extremality we should check  $2a^2 > 1$ . Since  $a > 0$  the last inequality is equivalent to  $\sqrt{2}a > 1$ . Denote

$$\gamma_1(\theta) = \sqrt{2}a - 1 = \sqrt{2} \frac{(\theta - 1)z_1}{(\theta + m - 1)z_1 + q - m} - 1,$$

where

$$z_1 = \left( \frac{\theta - 1 - \sqrt{(\theta - 1)^2 - 4m(q - m)}}{2m} \right)^2.$$

We have

$$\begin{aligned} \gamma_1'(\theta) &= \frac{\sqrt{2}(\theta - 1)z_1}{((\theta + m - 1)z_1 + q - m)^2} \left( \sqrt{z_1} - \frac{2(q - m)}{\sqrt{(\theta - 1)^2 - 4m(q - m)}} \right) \\ &= - \frac{\sqrt{2z_1}(\theta - 1)^2 z_1}{((\theta + m - 1)z_1 + q - m)^2 \sqrt{(\theta - 1)^2 - 4m(q - m)}}. \end{aligned}$$

Thus for any  $\theta \geq \theta_m$  we have  $\gamma_1'(\theta) < 0$ . Hence  $\gamma_1$  is a decreasing function of  $\theta$ . So to have  $\sqrt{2}a > 1$  it is necessary that

$$\gamma_1(\theta_m) = \sqrt{2} \frac{\sqrt{q - m}}{\sqrt{q - m} + \sqrt{m}} - 1 > 0,$$

which is satisfied for  $q \geq 7m$ . Moreover, we have  $\gamma_1(q + 1) = -(\frac{\sqrt{2}-1}{\sqrt{2}}) < 0$ . Consequently, there exists a unique  $\theta = \hat{\theta}$  such that  $\gamma_1(\hat{\theta}) = 0$ . To find  $\hat{\theta}$  we solve

$\gamma_1(\widehat{\theta}) = 0$  and by long computations we get

$$\widehat{\theta} = (\sqrt{2} - 1)q + 2m + 1.$$

Thus we proved that  $2a^2 > 1$  if  $\theta < \widehat{\theta}$ .

*Subcase:  $\theta \geq q + 1$ .* In this case by Lemma 2 we have  $z_1 < 1$  so we must check  $2b^2 > 1$ . Define

$$\xi_1(\theta) = \sqrt{2}b - 1 = \frac{(\theta - 1)\sqrt{2z_1}}{(\theta + m - 1)z_1 + q - m} - 1,$$

We have

$$\xi_1'(\theta) = \frac{\sqrt{2z_1}(\theta - 1)^2 z_1}{((\theta + m - 1)z_1 + q - m)^2 \sqrt{(\theta - 1)^2 - 4m(q - m)}} > 0.$$

Hence  $\xi_1$  is an increasing function of  $\theta$ . Moreover, we have  $\xi_1(q + 1) = \gamma_1(q + 1) = -(\frac{\sqrt{2}-1}{\sqrt{2}}) < 0$ . Since  $\xi_1(\theta)$  is increasing its maximal value should be at  $\theta \rightarrow +\infty$ . So we compute

$$\begin{aligned} \lim_{\theta \rightarrow \infty} \xi_1(\theta) &= -1 + \sqrt{2} \lim_{\theta \rightarrow \infty} \frac{(\theta - 1)\sqrt{z_1}}{(\theta + m - 1)z_1 + q - m} \\ &= -1 + \sqrt{2} \lim_{\theta \rightarrow \infty} \frac{1}{\left(\frac{\theta + m - 1}{\theta - 1}\right) \sqrt{z_1} + \frac{q - m}{(\theta - 1)\sqrt{z_1}}}. \end{aligned} \quad (4.4)$$

Using formula (2.10) we get

$$\begin{aligned} \lim_{\theta \rightarrow \infty} \sqrt{z_1} &= \lim_{\theta \rightarrow \infty} \frac{2(q - m)}{\theta - 1 + \sqrt{(\theta - 1)^2 - 4m(q - m)}} = 0. \\ \lim_{\theta \rightarrow \infty} (\theta - 1)\sqrt{z_1} &= \lim_{\theta \rightarrow \infty} \frac{2(q - m)}{1 + \sqrt{1 - \frac{4m(q - m)}{(\theta - 1)^2}}} = q - m. \end{aligned}$$

Using these formulas from (4.4) we get

$$\lim_{\theta \rightarrow \infty} \xi_1(\theta) = \sqrt{2} - 1 > 0.$$

Consequently, there exists a unique  $\theta = \theta^*$  such that  $\xi_1(\theta^*) = 0$ . To find  $\theta^*$  we solve  $\xi_1(\theta^*) = 0$  and by long computations we get

$$\theta^* = 1 + (\sqrt{2} + 1)q - 2m.$$

Thus we proved that  $2b^2 > 1$  if  $\theta > \theta^*$ .

**Case  $m = 1$ .** In this case we have to check  $\xi_1(\theta) = \sqrt{2}b - 1 > 0$ . We note that  $\xi_1'(\theta) > 0$  for each  $\theta \geq \theta_m$ , and

$$\xi_1(\theta_m) = \frac{\sqrt{2m}}{\sqrt{q - m} + \sqrt{m}} - 1 < 0.$$

Hence again we have  $\xi_1(\theta) > 0$  iff  $\theta \in (\theta^*, +\infty)$ .

(iii) **Case**  $m \geq 2$ . Since  $z_2 > 1$  for any  $\theta \geq \theta_m$ , we shall check only  $2a^2 > 1$ . Denote

$$\gamma_2(\theta) = \sqrt{2}a - 1 = \sqrt{2} \frac{(\theta - 1)z_2}{(\theta + m - 1)z_2 + q - m} - 1,$$

where

$$z_2 = \left( \frac{\theta - 1 + \sqrt{(\theta - 1)^2 - 4m(q - m)}}{2m} \right)^2.$$

We have

$$\gamma_2'(\theta) = \frac{\sqrt{2}(\theta - 1)z_2}{((\theta + m - 1)z_2 + q - m)^2} \left( \sqrt{z_2} + \frac{2(q - m)}{\sqrt{(\theta - 1)^2 - 4m(q - m)}} \right) > 0.$$

Thus for any  $\theta \geq \theta_m$  we have  $\gamma_2'(\theta) > 0$ . Hence  $\gamma_2$  is an increasing function of  $\theta$ . For  $q \geq 7m$  we have

$$\gamma_2(\theta_m) = \gamma_1(\theta_m) > 0.$$

Hence  $\gamma_2(\theta) > 0$  for all  $\theta > \theta_m$ . Consequently,  $\mu_2$  is non-extreme.

In case  $q \leq 6$  or  $q \geq 7$  with  $[q/7] < m$  we have  $\gamma_1(\theta_m) = \gamma_2(\theta_m) < 0$ . Since  $\gamma_2$  is an increasing function  $\gamma_2(\theta) = 0$  has a unique solution, which is equal to  $\hat{\theta}$ . Thus  $\gamma_2(\theta) > 0$  for all  $\theta > \hat{\theta}$ .

**Case**  $m = 1$ . Define

$$\xi_2(\theta) = \sqrt{2}b - 1 = \frac{(\theta - 1)\sqrt{2z_2}}{\theta z_2 + q - 1} - 1.$$

We have

$$\xi_2'(\theta) = -\frac{\sqrt{2z_2}(\theta - 1)^2 z_2}{(\theta z_2 + q - 1)^2 \sqrt{(\theta - 1)^2 - 4(q - 1)}} < 0.$$

Hence  $\xi_2$  is a decreasing function of  $\theta$ . Moreover, we have

$$\xi_2(\theta_m) = \xi_1(\theta_m) = -(1 - \frac{\sqrt{2}}{\sqrt{q - 1} + 1}) < 0.$$

Thus  $\xi_2(\theta) < 0$  for all  $\theta \geq \theta_m$ .

(iv) In this case the matrix  $T(\mathbb{P})$  is

$$T(\mathbb{P}) = \begin{pmatrix} \frac{(\theta+m-1)z}{Z_1} & \frac{q-m}{Z_1} \\ \frac{mz}{Z_2} & \frac{\theta+q-m-1}{Z_2} \end{pmatrix}, \quad (4.5)$$

which has eigenvalue  $\lambda_2(\hat{\mathbb{P}})$  given in (3.9). It is easy to check that  $\theta - 1 - (\sqrt{z_i} - 1)m \geq 0$ ,  $i = 1, 2$ , i.e.  $\lambda_2(\hat{\mathbb{P}}) > 0$ .

We define the following function

$$\eta_1(\theta) = \sqrt{2} \frac{(\theta - 1 - (\sqrt{z_1} - 1)m)z_1}{(\theta + m - 1)z_1 + q - m} - 1.$$

We have

$$\eta_1(\theta_m) = \eta_1(q+1) = -\frac{\sqrt{2}-1}{\sqrt{2}} < 0.$$

Consequently, by Rolle's theorem, there exists  $c \in (\theta_m, q+1)$  such that  $\eta'_1(c) = 0$ . Calculations show that such  $c$  is unique, and it corresponds to a maximum for  $\eta$ , i.e.  $\max_\theta \eta(\theta) = \eta(c)$ . Thus, for non-extremality, it is necessary to have  $\eta(c) > 0$ . One can see that this is satisfied if  $q$  is large enough, and a computer analysis shows that  $q \geq 85$ . In case  $\eta(c) > 0$  the equation  $\eta_1(\theta) = 0$  has exactly two solutions, which are denoted by  $\bar{\theta}, \bar{\bar{\theta}}$ .

By long computations one can see that  $\eta_1(\theta) > 0$  iff

$$(\theta-1)^3 - (\sqrt{2}-1)q(\theta-1)^2 - 2(2\sqrt{2}-1)m(q-m)(\theta-1) + 2qm(q-m) < 0. \quad (4.6)$$

Thus  $\bar{\theta}, \bar{\bar{\theta}}$  are roots of the equation (4.2).

This completes the proof.  $\square$

**Remark 1.** In case  $q = 7$  by  $q \geq 7m$  we only have  $m = 1$ . Then  $\theta_1 = 1 + 2\sqrt{6} \approx 5.898979$  and  $\hat{\theta} = 3 + 7(\sqrt{2}-1) \approx 5.899494$ . Hence  $\hat{\theta} - \theta_1 \approx 0,00051$ .

## 5. CONDITIONS FOR EXTREMALITY

We turn our attention to sufficient conditions for extremality (or non-reconstructability in information-theoretic language) of the full chains of the Potts model, depending on coupling strength parameterized by  $\theta$ , the block size  $m$  and the branch of the boundary law  $z$ . To do so, we use a result of [13] to establish a bound for reconstruction insolvability corresponding to the matrix (channel) of a solution  $z \neq 1$ . In [13] the key ingredients are two quantities,  $\kappa$  and  $\gamma$ , which bound the rates of percolation of disagreement down and up the tree, respectively. Both are properties of the collection of Gibbs measures  $\{\mu_{\mathcal{T}}^\tau\}$ , where the boundary condition  $\tau$  is fixed and  $\mathcal{T}$  ranges over all initial finite complete subtrees of  $\Gamma^k$ . For a given initial complete subtree  $\mathcal{T}$  of  $\Gamma^k$  and a vertex  $x \in \mathcal{T}$ , we write  $\mathcal{T}_x$  for the (maximal) subtree of  $\mathcal{T}$  rooted at  $x$ . When  $x$  is not the root of  $\mathcal{T}$ , let  $\mu_{\mathcal{T}_x}^s$  denote the Gibbs measure in which the parent of  $x$  has its spin fixed to  $s$  and the configuration on the bottom boundary of  $\mathcal{T}_x$  is specified by  $\tau$ .

For two measures  $\mu_1$  and  $\mu_2$  denote the variation distance by

$$\|\mu_1 - \mu_2\|_x = \frac{1}{2} \sum_{i=1}^q |\mu_1(\sigma(x) = i) - \mu_2(\sigma(x) = i)|.$$

Let  $\eta^{x,s}$  be the configuration  $\eta$  with the spin at  $x$  set to  $s$ .

Following [13] define

$$\begin{aligned} \kappa &\equiv \kappa(\mu) = \sup_{x \in \Gamma^k} \max_{x,s,s'} \|\mu_{\mathcal{T}_x}^s - \mu_{\mathcal{T}_x}^{s'}\|_x; \\ \gamma &\equiv \gamma(\mu) = \sup_{A \subset \Gamma^k} \max \|\mu_A^{\eta^{y,s}} - \mu_A^{\eta^{y,s'}}\|_x, \end{aligned}$$

where the maximum is taken over all boundary conditions  $\eta$ , all sites  $y \in \partial A$ , all neighbors  $x \in A$  of  $y$ , and all spins  $s, s' \in \{1, \dots, q\}$ .

As the main ingredient we apply [13, Theorem 9.3], which says that for an arbitrary channel  $\mathbb{P} = (P_{ij})_{i,j=1}^q$  on a tree reconstruction of the corresponding tree-indexed Markov chain (splitting Gibbs measure) is impossible if  $k\kappa\gamma < 1$ .

Note that  $\kappa$  has the particularly simple form

$$\kappa = \frac{1}{2} \max_{i,j} \sum_l |P_{il} - P_{jl}| \quad (5.1)$$

and  $\gamma$  is a constant which have not a clean general formula, but has a corresponding bound in specific models. For example, if  $\mathbb{P}$  is the symmetric channel of the Potts model (i.e. corresponding to solution  $z = 1$ ) then  $\gamma \leq \frac{\theta-1}{\theta+1}$  [13, Theorem 8.1].

Consider the case  $z \neq 1$  (where  $z = x^2$  and  $x$  is a solution to (2.7)), fix solution in the form  $(\underbrace{z, z, \dots, z}_m, 1, \dots, 1)$  and corresponding matrix  $\mathbb{P}$  given by (3.3).

The following proposition contains an essential part of our work. It generalizes [13, Proposition 8.2] from the case  $z = 1$  to the case  $z \neq 1$ . Moreover, we note that in case  $z \neq 1$  we have to treat the dependence on the additional blocksize-parameter  $m$ .

**Proposition 5.** *Assume  $m \leq [q/2]$ . Recall the matrix  $\mathbb{P}$ , given by (3.3), and denote by  $\mu = \mu(\theta, m)$  the corresponding Gibbs measure. Then, for any subset  $A \subset \mathcal{T}$ , any boundary configuration  $\eta$ , any pair of spins  $(s_1, s_2)$ , any site  $y \in \partial A$ , and any neighbor  $x \in A$  of  $y$ , we have*

$$\|\mu_A^{\eta^{y,s_1}} - \mu_A^{\eta^{y,s_2}}\|_x = \mathbb{K}(p(s_1), u(s_1); p(s_2), u(s_2)),$$

where  $p(s) = \mu_A^{\eta^{y, free}}(\sigma(x) = s)$ ,  $u(s) = \sum_{\substack{j=m+1 \\ j \neq s}}^q p(j)$  and the function  $\mathbb{K}$  is defined by

$$\mathbb{K}(p(s_1), u(s_1); p(s_2), u(s_2)) = \max\{p^{s_1}(s_1) - p^{s_2}(s_1), p^{s_2}(s_2) - p^{s_1}(s_2)\},$$

with  $p^t(s) = \mu_A^{\eta^{y,t}}(\sigma(x) = s)$ .

*Proof.* By definition of the matrix  $\mathbb{P}$  we have

$$p^t(s) = \frac{\exp(h_s + \beta \delta_{ts}) p(s)}{\sum_{l=1}^q \exp(h_l + \beta \delta_{tl}) p(l)} = \begin{cases} \frac{\theta^{\delta_{ts}} z p(s)}{(\theta-1)z p(t) + (1-z)u(t) + z}, & \text{if } t, s \leq m \\ \frac{p(s)}{(\theta-1)z p(t) + (1-z)u(t) + z}, & \text{if } t \leq m, s > m \\ \frac{z p(s)}{(\theta-z)p(t) + (1-z)u(t) + z}, & \text{if } t > m, s \leq m \\ \frac{\theta^{\delta_{ts}} p(s)}{(\theta-z)p(t) + (1-z)u(t) + z}, & \text{if } t, s \geq m+1. \end{cases} \quad (5.2)$$

The proposition follows from the following Lemma 3.  $\square$

**Lemma 3.** (i) If

$$p(s_1) \geq \begin{cases} p(s_2), & \text{for } s_1, s_2 \leq m \text{ and } s_1, s_2 > m \\ z^{-1}p(s_2), & \text{for } s_1 \leq m \text{ } s_2 > m \\ zp(s_2), & \text{for } s_1 > m \text{ } s_2 \leq m \end{cases}$$

then

- a)  $p^{s_1}(s) \leq p^{s_2}(s)$  for all  $s \neq s_1$ ;
- b)  $p^{s_1}(s_1) \geq p^{s_2}(s_1)$ ;
- c)  $p^{s_2}(s) - p^{s_1}(s) \leq p^{s_1}(s_1) - p^{s_2}(s_1)$ , for all  $s$ .

(ii) If

$$p(s_1) \leq \begin{cases} p(s_2), & \text{for } s_1, s_2 \leq m \text{ and } s_1, s_2 > m \\ z^{-1}p(s_2), & \text{for } s_1 \leq m \text{ } s_2 > m \\ zp(s_2), & \text{for } s_1 > m \text{ } s_2 \leq m \end{cases}$$

then

- a')  $p^{s_1}(s) \geq p^{s_2}(s)$  for all  $s \neq s_2$ ;
- b')  $p^{s_2}(s_2) \geq p^{s_1}(s_2)$ ;
- c')  $p^{s_1}(s) - p^{s_2}(s) \leq p^{s_2}(s_2) - p^{s_1}(s_2)$ , for all  $s$ .

*Proof.* (i).

a) Consider the following possible cases

Case:  $s_1, s_2 \leq m$ . Since  $s \neq s_1$  we should have

$$\begin{aligned} & p^{s_1}(s) - p^{s_2}(s) \\ = & \begin{cases} \frac{zp(s)}{(\theta-1)zp(s_1)+(1-z)u(s_1)+z} - \frac{\theta^{\delta s_2 s} zp(s)}{(\theta-1)zp(s_2)+(1-z)u(s_2)+z}, & \text{if } s \leq m \\ \frac{p(s)}{(\theta-1)zp(s_1)+(1-z)u(s_1)+z} - \frac{p(s)}{(\theta-1)zp(s_2)+(1-z)u(s_2)+z}, & \text{if } s > m \end{cases} \leq 0. \end{aligned} \quad (5.3)$$

Since  $\theta > 1$ , if we prove that the inequality (5.3) holds for  $s \neq s_2$  then it also true for  $s = s_2$ . In case  $s \neq s_2$  the inequality (5.3) is equivalent to

$$\begin{aligned} & (\theta-1)zp(s_2) + (1-z)u(s_2) + z - [(\theta-1)zp(s_1) + (1-z)u(s_1) + z] \\ = & (\theta-1)z(p(s_2) - p(s_1)) + (1-z)(u(s_2) - u(s_1)) = (\theta-1)z(p(s_2) - p(s_1)) \leq 0, \end{aligned}$$

here we used  $u(s_1) = u(s_2)$ , which holds by definition of the function  $u(s)$  and assumption  $s_1, s_2 \leq m$ .

Case:  $s_1 \leq m, s_2 > m$ . In this case for  $s \neq s_1$  we should prove the following

$$\begin{aligned} & p^{s_1}(s) - p^{s_2}(s) \\ = & \begin{cases} \frac{zp(s)}{(\theta-1)zp(s_1)+(1-z)u(s_1)+z} - \frac{zp(s)}{(\theta-z)p(s_2)+(1-z)u(s_2)+z}, & \text{if } s \leq m \\ \frac{p(s)}{(\theta-1)zp(s_1)+(1-z)u(s_1)+z} - \frac{\theta^{\delta s s_2} p(s)}{(\theta-z)p(s_2)+(1-z)u(s_2)+z}, & \text{if } s > m \end{cases} \leq 0. \end{aligned} \quad (5.4)$$

Similarly as in the previous case this inequality can be reduced to

$$\begin{aligned} & (\theta - z)p(s_2) + (1 - z)u(s_2) + z - [(\theta - 1)zp(s_1) + (1 - z)u(s_1) + z] \\ &= (\theta - z)p(s_2) - (\theta - 1)zp(s_1) + (1 - z)(u(s_2) - u(s_1)) \leq 0. \end{aligned} \quad (5.5)$$

For  $s_1 \leq m$  and  $s_2 > m$  we have  $u(s_2) - u(s_1) = -p(s_2)$ . Using this equality the inequality (5.5) can be written as

$$(\theta - 1)(p(s_2) - zp(s_1)) \leq 0.$$

*Case:*  $s_1 > m, s_2 \leq m$ . This case is very similar to the previous case and we get

$$(\theta - 1)(zp(s_2) - p(s_1)) \leq 0.$$

*Case:*  $s_1 > m, s_2 > m$ . For  $s \neq s_1$  we should prove the following

$$\begin{aligned} & p^{s_1}(s) - p^{s_2}(s) \\ &= \begin{cases} \frac{zp(s)}{(\theta - z)p(s_1) + (1 - z)u(s_1) + z} - \frac{zp(s)}{(\theta - z)p(s_2) + (1 - z)u(s_2) + z}, & \text{if } s \leq m \\ \frac{p(s)}{(\theta - z)p(s_1) + (1 - z)u(s_1) + z} - \frac{\theta^{\delta s s_2} p(s)}{(\theta - z)p(s_2) + (1 - z)u(s_2) + z}, & \text{if } s > m \end{cases} \leq 0 \end{aligned} \quad (5.6)$$

This can be reduced to

$$\begin{aligned} & (\theta - z)p(s_2) + (1 - z)u(s_2) + z - [(\theta - z)p(s_1) + (1 - z)u(s_1) + z] \\ &= (\theta - z)(p(s_2) - p(s_1)) + (1 - z)(u(s_2) - u(s_1)) \leq 0. \end{aligned} \quad (5.7)$$

For  $s_1 > m$  and  $s_2 > m$  we have  $u(s_2) - u(s_1) = p(s_1) - p(s_2)$ . Using this equality the inequality (5.7) can be written as

$$(\theta - 1)(p(s_2) - p(s_1)) \leq 0.$$

b) For each above-mentioned four cases the proof of the inequality  $p^{s_1}(s_1) \geq p^{s_2}(s_1)$  is the following

$$p^{s_1}(s_1) = 1 - \sum_{s \neq s_1} p^{s_1}(s) \geq 1 - \sum_{s \neq s_1} p^{s_2}(s) = p^{s_2}(s_1).$$

c) Using the results of a) and b) we get

$$\begin{aligned} p^{s_2}(s) - p^{s_1}(s) &= 1 - \sum_{t \neq s} p^{s_2}(t) - \left(1 - \sum_{t \neq s} p^{s_1}(t)\right) = \sum_{t \neq s} (p^{s_1}(t) - p^{s_2}(t)) \\ &= p^{s_1}(s_1) - p^{s_2}(s_1) + \sum_{t \neq s, s_1} (p^{s_1}(t) - p^{s_2}(t)) \leq p^{s_1}(s_1) - p^{s_2}(s_1). \end{aligned}$$

(ii). This is similar to the part (i).

□



**Lemma 4.** For any  $s, t \in \{1, \dots, q\}$  with  $s \neq t$  we have

$$p^s(s) - p^t(s) = \begin{cases} \frac{\theta zp(s)}{(\theta-1)zp(s)+(1-z)u(s)+z} - \frac{zp(s)}{(\theta-1)zp(t)+(1-z)u(s)+z}, & \text{if } s, t \leq m \\ \frac{\theta zp(s)}{(\theta-1)zp(s)+(1-z)u(s)+z} - \frac{zp(s)}{(\theta-1)p(t)+(1-z)u(s)+z}, & \text{if } s \leq m, t > m \\ \frac{\theta p(s)}{(\theta-z)p(s)+(1-z)u(s)+z} - \frac{p(s)}{(\theta-1)zp(t)+(1-z)(u(s)+p(s))+z}, & \text{if } s > m, t \leq m \\ \frac{\theta p(s)}{(\theta-z)p(s)+(1-z)u(s)+z} - \frac{p(s)}{(\theta-1)p(t)+(1-z)(u(s)+p(s))+z}, & \text{if } s, t > m \end{cases}$$

*Proof.* The proof simply follows from (5.2) by the formula

$$u(t) = \begin{cases} u(s) & \text{if } s, t \leq m \\ u(s) - p(t) & \text{if } s \leq m, t > m \\ u(s) + p(s) & \text{if } s > m, t \leq m \\ u(s) + p(s) - p(t) & \text{if } s, t > m. \end{cases}$$

□

Let  $p$  be a probability distribution on  $\{1, \dots, q\}$ . For  $p_1, p_2, u \geq 0$ ,  $p_1 + p_2 + u \leq 1$ , define the following functions

$$\begin{aligned} K_1(p_1, p_2, u) &= \frac{\theta zp_1}{(\theta-1)zp_1 + (1-z)u + z} - \frac{zp_1}{(\theta-1)zp_2 + (1-z)u + z}; \\ K_2(p_1, p_2, u) &= \frac{\theta zp_1}{(\theta-1)zp_1 + (1-z)u + z} - \frac{zp_1}{(\theta-1)p_2 + (1-z)u + z}; \\ K_3(p_1, p_2, u) &= \frac{\theta p_1}{(\theta-z)p_1 + (1-z)u + z} - \frac{p_1}{(\theta-1)zp_2 + (1-z)(u + p_1) + z}; \\ K_4(p_1, p_2, u) &= \frac{\theta p_1}{(\theta-z)p_1 + (1-z)u + z} - \frac{p_1}{(\theta-1)p_2 + (1-z)(u + p_1) + z}. \end{aligned}$$

**Lemma 5.** We have

$$K_1(p_1, p_2, u) \leq \frac{\theta-1}{\theta+1} \quad \text{and} \quad K_3(p_1, p_2, u) \leq \frac{\theta-1}{\theta+1}.$$

*Proof.* First let  $p_1 + p_2 = v$ ,  $0 \leq v \leq 1$ . We compute  $\max_{\substack{p_1 \geq 0, 0 \leq v, u \leq 1, \\ u+v \leq 1}} K_1(p_1, v - p_1, u)$ . Note that  $K_1(p, v - p, u)$  is an increasing function of  $v$ . Consequently,

$$\max_{\substack{p \geq 0, 0 \leq v, u \leq 1, \\ u+v \leq 1}} K_1(p, v - p, u) \leq K_1(p, 1 - u - p, u) = \zeta(p, u).$$

Thus we should prove that  $\zeta(p, u) \leq \frac{\theta-1}{\theta+1}$ . But this inequality is equivalent to

$$\theta z^2(2p + u - 1)^2 + [(\theta + 1)z(1 - u - p) + u]u \geq 0,$$

which is true for any  $p, u$  with  $p + u \leq 1$ .

Similarly, one can get

$$\max_{\substack{p_1, p_2, u \geq 0: \\ p_1 + p_2 + u \leq 1}} K_3(p_1, p_2, u) \leq K_3(p_1, 1 - u - p_1, u) = g(p_1, u).$$

Note that  $g(p, u) \leq \frac{\theta-1}{\theta+1}$  is equivalent to

$$[\{p - z(1 - p - u)\}^2 + z(1 - p - u)u]\theta + u(z(1 - p - u) + u) \geq 0,$$

which is true for any  $p, u$  with  $p + u \leq 1$ .  $\square$

The following proposition gives a bound for  $\gamma$ .

**Proposition 6.** 1) If  $z \geq 1$  then

$$\gamma \leq \max_{\substack{p_1, p_2, u \geq 0: \\ p_1 + p_2 + u \leq 1}} \{K_1(p_1, p_2, u), K_3(p_1, p_2, u)\} \leq \frac{\theta - 1}{\theta + 1}. \quad (5.8)$$

2) If  $z \leq 1$  then

$$\gamma \leq \max_{\substack{p_1, p_2, u \geq 0: \\ p_1 + p_2 + u \leq 1}} \{K_2(p_1, p_2, u), K_4(p_1, p_2, u)\} \leq \frac{\theta - 1}{\theta + 1} + 1 - z. \quad (5.9)$$

*Proof.* 1) For  $z \geq 1$  it is easy to see that  $K_2(p_1, p_2, u) \leq K_1(p_1, p_2, u)$  and  $K_4(p_1, p_2, u) \leq K_3(p_1, p_2, u)$ . Consequently applying Lemma 5 we get 1).

2) For  $z \leq 1$  it is easy to see that  $K_2(p_1, p_2, u) \geq K_1(p_1, p_2, u)$  and  $K_4(p_1, p_2, u) \geq K_3(p_1, p_2, u)$ . Denote

$$\begin{aligned} V(p_1, p_2, u) &= K_2(p_1, p_2, u) - K_1(p_1, p_2, u), \\ W(p_1, p_2, u) &= K_4(p_1, p_2, u) - K_3(p_1, p_2, u). \end{aligned}$$

We have

$$\begin{aligned} V(p_1, p_2, u) &= \frac{(1 - z)z(\theta - 1)p_1p_2}{((\theta - 1)zp_2 + (1 - z)u + z)((\theta - 1)p_2 + (1 - z)u + z)} \\ &\leq \frac{(1 - z)zp_1}{(\theta - 1)zp_2 + (1 - z)u + z} \leq \frac{(1 - z)z(1 - p_2 - u)}{(\theta - 1)zp_2 + (1 - z)u + z} \equiv (1 - z)z\mathcal{V}(p_2, u). \end{aligned}$$

It is easy to see that  $\mathcal{V}(p, u) \leq \mathcal{V}(0, 0) = 1/z$ . Thus  $V(p_1, p_2, u) \leq 1 - z$ .

For  $W$  we have

$$\begin{aligned} W(p_1, p_2, u) &= \frac{(1 - z)(\theta - 1)p_1p_2}{((\theta - 1)zp_2 + (1 - z)(u + p_1) + z)((\theta - 1)p_2 + (1 - z)(u + p_1) + z)} \\ &\leq \frac{(1 - z)p_1}{(\theta - 1)zp_2 + (1 - z)(u + p_1) + z} \\ &\leq \frac{(1 - z)p_1}{(1 - z)(u + p_1) + z} \equiv (1 - z)\mathcal{W}(p_1, u). \end{aligned}$$

It is easy to see that  $\mathcal{W}$  is an increasing function of  $p_1$ . Consequently

$$\mathcal{W}(p_1, u) \leq \mathcal{W}(1 - u, u) = 1 - u \leq 1.$$

Thus  $W(p_1, p_2, u) \leq 1 - z$ . Now using Lemma 5 we get 2).  $\square$

Having done the more difficult job to estimate  $\gamma$  we shall now compute the remaining constant  $\kappa$ . Using (5.1) and (3.3) for  $i \neq j$  we get

$$\frac{1}{2} \sum_{l=1}^q |P_{il} - P_{jl}| = \begin{cases} a, & \text{if } i, j = 1, \dots, m \\ b, & \text{if } i, j = m+1, \dots, q \\ c, & \text{otherwise,} \end{cases}$$

where  $a$  and  $b$  are defined in (3.9) and

$$c = \frac{1}{2Z_1} \left( z|\theta - \sqrt[k]{z}| + |1 - \theta \sqrt[k]{z}| + (z(m-1) + q - m - 1)|1 - \sqrt[k]{z}| \right).$$

Hence we arrive at

$$\kappa = \begin{cases} \max\{b, c\}, & \text{if } m = 1 \\ \max\{a, b, c\} & \text{if } m \geq 2. \end{cases} \quad (5.10)$$

**Case:** Let us specialize to the binary tree  $k = 2$  and  $m = 1$ . In this special case we shall apply the condition  $2\gamma\kappa < 1$  together with our upper bounds for  $\gamma$  and  $\kappa$ . This gives the following result:

**Theorem 5.** *If  $k = 2$ ,  $m = 1$  then*

- (a) *There exists  $\theta^{**} > \theta_c = q + 1$  such that the measure  $\mu_1(\theta, 1)$  is extreme for any  $\theta \in [1 + 2\sqrt{q-1}, \theta^{**})$ ,  $q \geq 2$ .*
- (b) *The measure  $\mu_2(\theta, 1)$  is extreme for any  $\theta \geq 1 + 2\sqrt{q-1}$ ,  $q \geq 2$ . (see Fig.5)*

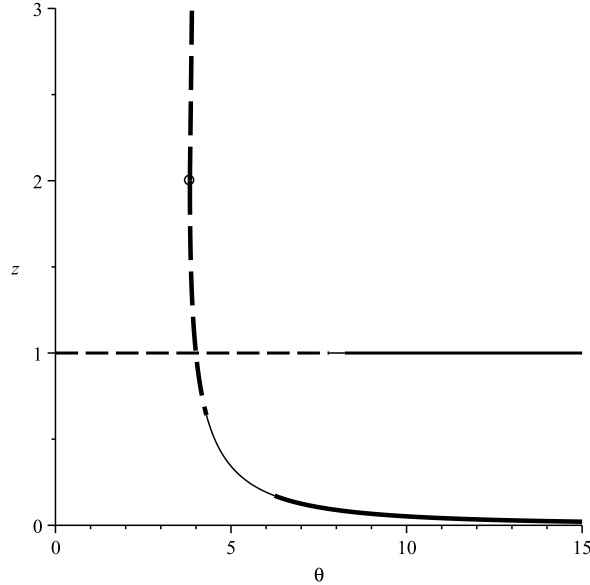


Fig. 5. The graphs of functions  $z_i = z_i(m, \theta)$ , for  $q = 3$ ,  $m = 1$  and the graph of  $z(\theta) \equiv 1$ . The bold curves correspond to regions of solutions where corresponding TISGM is non-extreme. The dashed

bold curves correspond to regions of solutions where corresponding TISGM is extreme. The gaps between the two types of curves are given by thin curves.

*Proof.* Since  $z_i$  is a solution to  $z_i - (\theta - 1)\sqrt{z_i} + q - 1 = 0$  and  $\theta > \sqrt{z_i}$ ,  $i = 1, 2$  we can simplify the expression of  $c$ . For  $z_i \geq 1$  we get

$$c = \frac{1}{Z_1} ((\theta + q - 2)\sqrt{z_i} - (q - 1)) \quad (5.11)$$

and  $b < c$ . Consequently,  $\kappa = c$ .

(a) *Case:*  $z_1 \geq 1$ . For  $\theta \in [\theta_1, \theta_c] = [1 + 2\sqrt{q-1}, q+1]$  we have  $z_1 \geq 1$ . Hence we should check

$$2\gamma\kappa \leq 2c \frac{\theta - 1}{\theta + 1} = \frac{2(\theta - 1)((\theta + q - 2)\sqrt{z_1} - (q - 1))}{(\theta z_1 + q - 1)(\theta + 1)} < 1. \quad (5.12)$$

Denoting

$$\psi_1(\theta) = (\theta - 1)(\theta^2 - \theta + 6 - 4q) - (\theta^2 - \theta + 4 - 2q)\sqrt{(\theta - 1)^2 - 4(q - 1)}$$

it is easy to see that the inequality (5.12) is equivalent to  $\psi_1(\theta) > 0$ ,  $\theta \in [\theta_1, \theta_c]$ . The inequality  $\psi_1(\theta) > 0$  can be reduced to

$$u(\theta) = \theta^3 - (q - 1)\theta^2 - (2q - 3)\theta + (4q^2 - 13q + 11) > 0, \quad \theta \in [\theta_1, \theta_c]. \quad (5.13)$$

The function  $u(\theta)$  for  $\theta \in (0, \theta_c)$  has minimum at  $\bar{\theta} = \bar{\theta}(q) = (q - 1 + \sqrt{q^2 + 4q - 8})/3$ . Moreover  $u(\bar{\theta})$  is an increasing function of  $q \geq 2$  with  $u(\bar{\theta}(2)) = u(1) > 0$ . Consequently,  $\psi_1(\theta) > 0$  and  $\mu_1(\theta, 1)$  is extreme for all  $\theta \in [1 + 2\sqrt{q-1}, q+1]$ .

*Case:*  $z_1 < 1$ . In this case independently on  $q \geq 2$  we get

$$c = \frac{1}{Z_1} (\theta\sqrt{z_1} - 1).$$

Using  $z_1 \leq 1$  (i.e.  $\theta \in [\theta_c, +\infty)$ ) we get  $b \geq c$ . Consequently,  $\kappa = b$ . Hence by Proposition 6 we should check

$$2\gamma\kappa \leq 2b \left( \frac{\theta - 1}{\theta + 1} + 1 - z \right) < 1. \quad (5.14)$$

Since  $z_1$  is a solution to  $z_1 - (\theta - 1)\sqrt{z_1} + q - 1 = 0$  the inequality (5.14) can be written as

$$(2\theta^3 - \theta^2 - (2q + 3)\theta - 2(q - 2))\sqrt{z_1} - (q - 1)(\theta + 1)(2\theta - 1) > 0$$

i.e.

$$\begin{aligned} f(\theta) &= 2\theta^4 - 3\theta^3 - 2(3q - 1)\theta^2 - (2q - 9)\theta + 2(2q - 3) \\ &\quad - (2\theta^3 - \theta^2 - (2q + 3)\theta - 2(q - 2))\sqrt{(\theta - 1)^2 - 4(q - 1)} > 0. \end{aligned}$$

Simplifying the last inequality we obtain

$$s(\theta) = 6\theta^4 - (2q + 3)\theta^3 - (4q^2 + 7q + 3)\theta^2 - (8q^2 - 8q - 11)\theta - (4q^2 - 13q + 11) < 0.$$

It is well known (see [17, p.28]) that the number of positive roots of a polynomial does not exceed the number of sign changes of its coefficients. Using this fact for  $s(\theta)$  we see that the equation  $s(\theta) = 0$  has up to one positive solution. Moreover,

for any  $q \geq 2$  we have  $s(q+1) < 0$  and  $s(\theta^*) > 0$ , where  $\theta^*$  is defined in (4.1). So there exists a unique  $\theta^{**}$  with  $q+1 < \theta^{**} < \theta^*$  such that  $s(\theta^{**}) = 0$ . Thus  $f(\theta) > 0$  and  $\mu_1(\theta, 1)$  is extreme for  $\theta \in [q+1, \theta^{**})$ .

For  $q = 3$  a numerical analysis shows that  $\theta^{**} \approx 4.2277$ . So  $\theta^* - \theta^{**} = 2.0149$  (see Fig.6). This is the length of the gap between Kesten-Stigum threshold beyond which non-extremality certainly holds and our bound below which extremality holds.

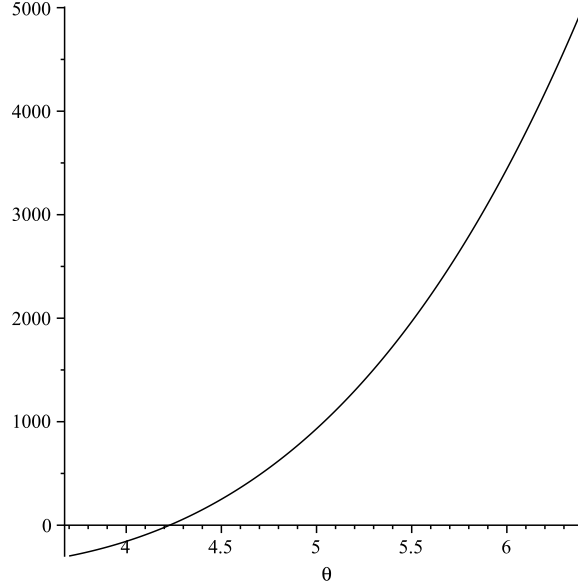


Fig. 6. The graph of the function  $s(\theta)$ , for  $q = 3$ . The solution of  $s(\theta^{**}) = 0$  is  $\theta^{**} \approx 4.2277$ .

(b) For  $\theta \geq 1 + 2\sqrt{q-1}$  we have  $z_2 > 1$ , i.e  $z_2$  is larger than 1 as soon as it exists. Similarly as the case (a) we shall check  $2\gamma\kappa \leq 2c\frac{\theta-1}{\theta+1} < 1$ . Denoting

$$\psi_2(\theta) = (\theta-1)(\theta^2 - \theta + 6 - 4q) + (\theta^2 - \theta + 4 - 2q)\sqrt{(\theta-1)^2 - 4(q-1)}$$

it is easy to see that the last inequality is equivalent to  $\psi_2(\theta) > 0$ ,  $\theta \geq 1 + 2\sqrt{q-1}$ . Note that

$$\theta^2 - \theta + 6 - 4q \leq \theta^2 - \theta + 4 - 2q,$$

so if we prove that

$$\theta^2 - \theta + 6 - 4q > 0 \tag{5.15}$$

then it follows that  $\psi_2(\theta) > 0$ . The solutions of the inequality (5.15) is  $\theta > (1 + \sqrt{16q-23})/2$ . Moreover, one can easily check that

$$(1 + \sqrt{16q-23})/2 < 1 + 2\sqrt{q-1} = \theta_1$$

Thus we proved that  $\psi_2(\theta) > 0$  and  $\mu_2(\theta, 1)$  is extreme for any  $\theta \geq 1 + 2\sqrt{q-1}$ .  $\square$

**Remark 2.** In this section we are considering the solution of the form  $(\underbrace{z, z, \dots, z}_m, 1, \dots, 1)$ ,

but of course the results are true for any permutations of the coordinates of this solution. We note that  $\mu_2(\theta, 1)$  corresponds to  $m = 1$ , by the permutations we get  $q$  distinct measures similar to  $\mu_2(\theta, 1)$ . The extremality of such measures was first proved in [3]. Our proof is an alternative to the proof of [3]. All other extremal TISGMs constructed in our paper are new.

**Case:**  $k = 2$  (binary tree),  $q < 7m$ ,  $m \geq 2$ . From condition  $2 \leq m \leq [q/2]$  it follows that  $q \geq 4$ .

- Theorem 6.** (i) If  $m = 2$  then for each  $q = 4, 5, 6, 7, 8$  there exists  $\check{\theta} > q + 1$  such that the measure  $\mu_1(\theta, 2)$  is extreme for any  $\theta \in [\theta_2, \check{\theta})$ . Moreover, if  $q = 9, 10, 11, 12, 13$  then there exists  $\acute{\theta} = \acute{\theta}(q)$  such that  $\theta_2 < \acute{\theta} < q + 1$  and  $\mu_1(\theta, 2)$  is extreme for  $\theta \in [\acute{\theta}, \check{\theta})$ .
- (ii) If  $m = 2$  then for each  $q = 4, 5, 6, 7, 8$  there exists  $\grave{\theta} = \grave{\theta}(q)$  such that  $\theta_2 < \grave{\theta} \leq q + 1$  and  $\mu_2(\theta, 2)$  is extreme for  $\theta \in [\theta_2, \grave{\theta})$  (see Fig. 7).
- (iii) If  $q < \frac{m+1}{2m} [3m + 1 + \sqrt{m^2 + 6m + 1}]$ ,  $m \geq 2$  then the measure  $\mu_1(\theta_m, m) = \mu_2(\theta_m, m)$  is extreme.

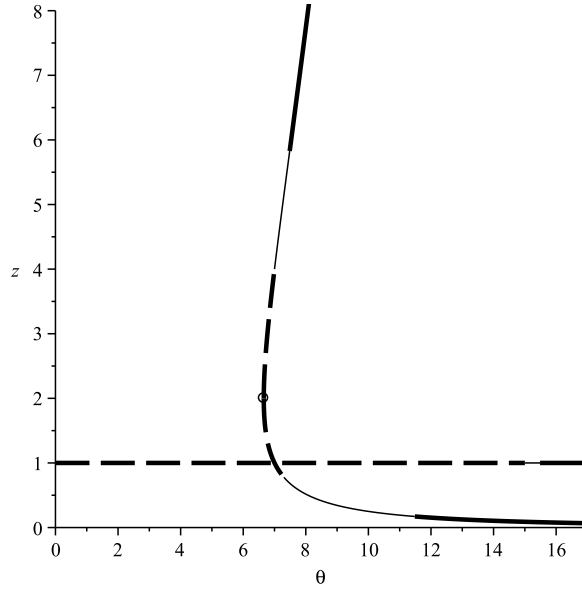


Fig. 7. The graphs of the functions  $z_i = z_i(m, \theta)$ , for  $q = 6$ ,  $m = 2$  and the graph of  $z(\theta) \equiv 1$ . The types of curves corresponding to certain extremality and certain non-extremality are as in Fig. 5.

*Proof.* Since  $z_i$  is a solution to  $mz - (\theta - 1)\sqrt{z} + q - m = 0$  for  $z = z_i \geq 1$ ,  $i = 1, 2$  we obtain

$$c = \frac{1}{m^2 Z_1} \left( \{(m-1)(\theta-1)^2 + m(q-m)\}\sqrt{z} - (m-1)(q-m)(\theta-1) \right).$$

We have  $a \geq c$ . Indeed,  $a \geq c$  is equivalent to

$$[(\theta-1)^2 - m(q-m)]\sqrt{z} - (q-m)(\theta-1) \geq 0,$$

i.e.

$$(\theta-1)[(\theta-1)^2 - 3m(q-m)] + [(\theta-1)^2 - m(q-m)]\sqrt{(\theta-1)^2 - 4m(q-m)} \geq 0.$$

This inequality is true for any  $\theta \geq \theta_m = 1 + 2\sqrt{m(q-m)}$ .

(i) *Case:*  $z_1 \geq 1$ . If  $z = z_1 \geq 1$ , i.e.  $\theta \in [\theta_m, \theta_c]$  then as shown above we have  $\kappa = a$ . Hence we should check

$$2\gamma\kappa \leq 2a \frac{\theta-1}{\theta+1} = \frac{2(\theta-1)^2 z_1}{((\theta+m-1)z_1 + (q-m))(\theta+1)} < 1. \quad (5.16)$$

The last inequality can be written as

$$[-\theta^2 + (m+4)\theta + m-3]z_1 + (\theta+1)(q-m) > 0. \quad (5.17)$$

Using

$$\sqrt{z_1} = \frac{\theta-1 - \sqrt{(\theta-1)^2 - 4m(q-m)}}{2m} = \frac{2(q-m)}{\theta-1 + \sqrt{(\theta-1)^2 - 4m(q-m)}}$$

from (5.17) we obtain

$$\theta^2 - (2m+4)\theta - 2m+3 - (\theta-3)\sqrt{(\theta-1)^2 - 4m(q-m)} < 0. \quad (5.18)$$

Now for  $m = 2$  and a given  $q < 14$  from (5.18) we obtain the assertions of (i) corresponding to the case  $\theta \leq \theta_c = q+1$ .

*Case:*  $z_1 < 1$ . In this case we have, independently on the values of  $q$  and  $m$ ,

$$c = \frac{1}{Z_1} (\theta\sqrt{z_1} - 1).$$

It is easy to see that  $b \geq c$  for  $z_1 < 1$ , i.e. for  $\theta > \theta_c$ . Consequently,  $\kappa = b$  and we should check

$$2\gamma\kappa \leq 2b \left( \frac{\theta-1}{\theta+1} + 1 - z_1 \right) = \frac{2(\theta-1)[2\theta - (\theta+1)z]\sqrt{z_1}}{(\theta+1)[(\theta+m-1)z_1 + q-m]} < 1. \quad (5.19)$$

Using  $mz_1 - (\theta-1)\sqrt{z_1} + q - m = 0$  for  $m = 2$  the inequality (5.19) can be written as

$$(\theta^3 - (2q+3)\theta - 2q+6)\sqrt{z_1} - (q-2)(\theta+1)\theta > 0$$

i.e.

$$\begin{aligned} \Theta(\theta, q) &= \theta^4 - \theta^3 - (6q-5)\theta^2 - (4q-17)\theta + 2q-6 \\ &\quad - (\theta^3 - (2q+3)\theta - 2q+6)\sqrt{(\theta-1)^2 - 8(q-2)} > 0. \end{aligned}$$

For each  $q = 4, \dots, 14$  the function  $\Theta(\theta, q)$  is a decreasing function on  $[\theta_c, +\infty)$ . Moreover  $\Theta(\theta_c, q) > 0$  and  $\Theta(\theta^*, q) < 0$ , where  $\theta^*$  is defined in (4.1). Consequently, there exists a unique  $\check{\theta}$  with  $\theta_c < \check{\theta} < \theta^*$  such that  $\Theta(\check{\theta}, q) = 0$ . Thus  $\Theta(\theta, q) > 0$  and  $\mu_1(\theta, 2)$  is extreme for  $\theta \in [\theta_c, \check{\theta})$ .

Note that for  $q = 6$  we have  $\theta^* = 3(1 + 2\sqrt{2}) \approx 11.485$  and a numerical analysis shows that  $\check{\theta} \approx 7.25$  (see Fig.8). So  $\theta^* - \check{\theta} \approx 4.23$ .

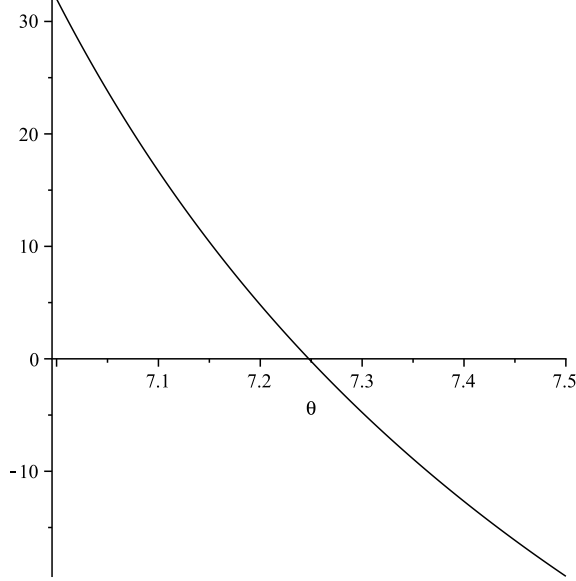


Fig. 8. The graph of the functions  $\Theta(\theta, q)$ , for  $q = 6$ .

(ii) Note that  $z = z_2 > 1$ , for each  $\theta \in [1 + 2\sqrt{m(q-m)}, +\infty)$ . Similarly as above for (5.18) we have to check

$$\theta^2 - (2m + 4)\theta - 2m + 3 + (\theta - 3)\sqrt{(\theta - 1)^2 - 4m(q - m)} < 0. \quad (5.20)$$

For  $m = 2$  and a given  $q < 9$ , from (5.20) we then obtain the assertions of (ii).

In the case  $q = 6$ ,  $m = 2$  it is easy to see that the LHS of (5.20) is zero for  $\theta = \check{\theta} = 7$  and the inequality is true for any  $\theta \in [\theta_2, 7)$ . In Theorem 4 we proved that  $\mu_2(\theta, m)$  is non-extreme if  $\theta > \hat{\theta}$  where  $\hat{\theta}$  is defined in (4.1) which now is  $\hat{\theta} = 6\sqrt{2} - 1 \approx 7.4852$ . So  $\hat{\theta} - 7 \approx 0.4852$ . This is the length of the gap between the Kesten-Stigum bound of non-extremality and our bound of extremality.

(iii) For  $\theta = \theta_m$  we have  $z_1 = z_2 > 1$  and from (5.18) we get

$$\theta_m^2 - (2m + 4)\theta_m - 2m + 3 < 0. \quad (5.21)$$



Using formula  $\theta_m = 1 + 2\sqrt{m(q-m)}$  from (5.21) we obtain

$$q < \frac{m+1}{2m} \left[ 3m+1 + \sqrt{m^2 + 6m + 1} \right], \quad m \geq 2. \quad (5.22)$$

This completes the proof.  $\square$

Let us conclude the paper with softer results which are valid for  $\theta$  close to the critical value  $\theta_c$  at which the lower branches of the boundary law  $z$  degenerate into the free value  $z = 1$  and the corresponding Markov chains become close to the free chain.

**Theorem 7.** a) For each  $m \leq [q/2]$  there exists a neighborhood  $U_m(\theta_c)$  of  $\theta_c$  such that the measure  $\mu_1(\theta, m)$  is extreme if  $\theta \in U_m(\theta_c)$  (see Fig.9).  
b) If  $m = 1$  or (5.22) is satisfied then there exists a neighborhood  $V_m(\theta_m)$  of  $\theta_m$  such that measures  $\mu_i(\theta, m)$ ,  $i = 1, 2$  are extreme if  $\theta \in V_m(\theta_m)$ . Moreover, for  $m = 1$  the measure  $\mu_2(\theta, 1)$  is always extreme.

*Proof.* a) As it was shown above  $\gamma$  and  $\kappa$  are bounded by continuous functions of  $\theta$ . We know that for  $\theta = \theta_c$  the measure  $\mu_0$  is extreme that is  $2\gamma\kappa \leq 2a_{\frac{\theta_c-1}{\theta_c+1}} < 1$ . Since  $\mu_1(\theta_c, m) = \mu_0$ , by the above-mentioned continuity in a sufficiently small neighborhood of  $\theta_c$  the measure  $\mu_1(\theta, m)$  is extreme.

b) This follows from Theorem 6 using the continuity similarly as in case a).  $\square$

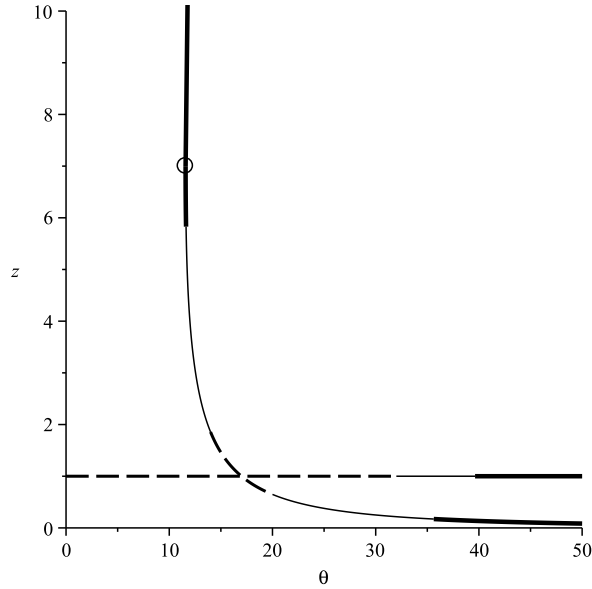


Fig. 9. The graphs of the functions  $z_i = z_i(m, \theta)$ , for  $q = 16$ ,  $m = 2$  and the graph of  $z(\theta) \equiv 1$ . The types of curves are again as in Fig.5.

**Theorem 8.** *For the ferromagnetic ( $\theta > 1$ ),  $q$ -state Potts model (with  $q \geq 3$ ) on the Cayley tree of order two, there exists a punctured neighborhood  $U(\theta_c)$  of  $\theta_c$  (i.e. without  $\theta_c$ ) such that there are at least  $2^{q-1} + q$  extreme TISGMs for each  $\theta \in U(\theta_c)$ .*

*Proof.* By Proposition 1 we know that for  $\theta_{[q/2]} < \theta \neq \theta_c$  there are  $2^q - 1$  TISGMs. One of them corresponds to  $z = 1$ . Half of remaining  $2^q - 2$ , i.e.  $2^{q-1} - 1$ , TISGMs generated by  $z_1$  and other  $2^{q-1} - 1$  TISGMs are generated by  $z_2$ . By above-mentioned results we know that  $\mu_2(\theta, 1)$  is extreme. The number of such measures is  $q$ . By Theorem 7, in a sufficiently small punctured neighborhood of  $\theta_c$  all measures  $\mu_1(\theta, m)$  are extreme. Thus the total number of extremal TISGM is at least  $2^{q-1} + q$ .  $\square$

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#### REFERENCES

- [1] A.C.D. van Enter, R. Fernández, A.D. Sokal, *Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory*, J. Stat. Phys. **72** (1993), 879–1167
- [2] M. Formentin, C. Külske, *A symmetric entropy bound on the non-reconstruction regime of Markov chains on Galton-Watson trees*. Electron. Commun. Probab. **14** (2009), 587–596.
- [3] N.N. Ganikhodjaev, *On pure phases of the three-state ferromagnetic Potts model on the second-order Bethe lattice*. Theor. Math. Phys. **85**(2) (1990), 1125–1134.
- [4] N.N. Ganikhodjaev, *On pure phases of the ferromagnetic Potts model Bethe lattices*, Dokl. AN Uzbekistan. No. 6-7 (1992), 4–7.
- [5] H.O. Georgii, *Gibbs Measures and Phase Transitions*, Second edition. de Gruyter Studies in Mathematics, 9. Walter de Gruyter, Berlin, 2011.
- [6] O. Häggström, *The random-cluster model on a homogeneous tree*, Probab. Theory and Relat. Fields **104** (1996) 231–253.
- [7] O. Häggström, *Is the fuzzy Potts model Gibbsian?* Ann. Inst. H. Poincaré Probab. Statist. **39** (2003), 891–917.
- [8] H. Kesten, B.P. Stigum, *Additional limit theorem for indecomposable multi-dimensional Galton-Watson processes*, Ann. Math. Statist. **37** (1966), 1463–1481.
- [9] O. Häggström, C. Külske, *Gibbs properties of the fuzzy Potts model on trees and in mean field*, Markov Proc. Rel. Fields **10**(3) (2004), 477–506.
- [10] C. Külske, U.A. Rozikov, R.M. Khakimov, *Description of all translation-invariant (splitting) Gibbs measures for the Potts model on a Cayley tree*, arXiv:1310.6220 [math-ph].
- [11] C. Külske, *Regularity properties of potentials for joint measures of random spin systems*, Markov Proc. Rel. Fields **10** (1) (2004) 75–88.
- [12] J.B. Martin, *Reconstruction thresholds on regular trees*. Discrete random walks (Paris, 2003), 191–204 (electronic), Discrete Math. Theor. Comput. Sci. Proc., AC, Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2003.
- [13] F. Martinelli, A. Sinclair, D. Weitz, *Fast mixing for independent sets, coloring and other models on trees*. Random Structures and Algorithms, **31** (2007), 134–172.

- [14] M. Mézard, A. Montanari, *Reconstruction on trees and spin glass transition*. J. Stat. Phys. **124** (2006), 1317–1350.
- [15] E. Mossel, Y. Peres, *Information flow on trees*, Ann. Appl. Probab. **13**(3) (2003), 817–844.
- [16] E. Mossel, *Survey: Information Flow on Trees*. Graphs, morphisms and statistical physics, 155–170, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 63, Amer. Math. Soc., Providence, RI, 2004.
- [17] V.V. Prasolov, *Polynomials* (Springer-Verlag Berlin Heidelberg, 2004)
- [18] U.A. Rozikov, *Gibbs measures on Cayley trees*. World Sci. Publ. Singapore. 2013.
- [19] A. Sly, *Reconstruction for the Potts model*. Ann. Probab. **39** (2011), 1365–1406.

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